3. ON A NEW THEOREM IN ELASTICITY

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1. The Equations of motion of an elastic system are[1]

$$\left.\begin{aligned}\rho\ddot{u} &= \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\\ \rho\ddot{v} &= \rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}\\ \rho\ddot{w} &= \rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z}\end{aligned}\right\} \qquad (1)$$

Multiplying the equations by u, v and w, and adding, we have,

$$\because\ u\ddot{u}=\tfrac{1}{2}\frac{d^2}{dt^2}(u^2)-\dot{u}^2,$$

$$\frac{\rho}{2}\frac{d^2}{dt^2}(u^2+v^2+w^2)-\rho(\dot{u}^2+\dot{v}^2+\dot{w}^2)$$

$$=\rho(Xu+Yv+Zw)+u\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial z}\right)$$

$$+v\left(\frac{\partial Y_x}{\partial x}+\frac{\partial Y_y}{\partial y}+\frac{\partial Y_z}{\partial z}\right)+w\left(\frac{\partial Z_x}{\partial x}+\frac{\partial Z_y}{\partial y}+\frac{\partial Z_z}{\partial z}\right) \qquad (2)$$

Now multiplying by $(dx \,.\, dy \,.\, dz \,.\, dt)$, and integrating we have,

$$\because \text{ since } \iiint u\left(\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}+\frac{\partial X_z}{\partial x}\right)dx\,dy\,dz=\iint uX_n dS$$

$$-\iiint\left(X_x\frac{du}{dx}+X_y\frac{du}{dy}+X_z\frac{du}{dz}\right)d\Omega.$$

$$\iiiint_{t=o}^{t=\tau}\frac{\rho}{2}\frac{d^2}{dt^2}(u^2+v^2+w^2)\,dt\,d\Omega-\iiiint_{t=o}^{t=\tau}2T\,dt\,d\Omega$$

$$=\iiiint_{t=o}^{t=\tau}\rho\,(Xu+Yv+Zw)\,dt\,.\,d\Omega$$

$$+\iiint_{t=o}^{t=\tau}(X_nu+Y_nv+Z_nw)dS\,dt-\iiiint_{t=o}^{t=\tau}2W\,d\Omega\,dt,$$

where

$$2W=X_xe_{xx}+Y_ye_{yy}+Z_ze_{zz}+X_ye_{xy}+X_ze_{xz}+Y_ze_{yz} \qquad (3)$$

Denoting by $\overline{T}$ the time-average of kinetic energy per unit volume, and by $\overline{W}$ the time-average of the potential energy per unit volume, we have

$$\iiint(\overline{W}-\overline{T})\,d\Omega=\frac{1}{2\tau}\iiiint_{t=o}^{t=\tau}\rho\,(Xu+Yv+Zw)\,dt\,.\,d\Omega$$

$$+\frac{1}{2\tau}\iiint_{t=o}^{t=\tau}\rho\,(X_nu+Y_nv+Z_nw)\,dS\,dt$$

$$-\frac{1}{\tau}\iiint\left[\frac{\rho}{4}\frac{d}{dt^2}(u^2+v^2+w^2)\right]_{t=o}^{t=\tau}d\Omega \qquad (4)$$

2. If we now take a closed volume Ω and $\overline{W}$, $\overline{T}$ denote the average values over time as well as over space, we shall have

$$\overline{W}-\overline{T}=\frac{1}{2\Omega\tau}\iiiint_{t=o}^{t=\tau}\rho\,(Xu+Yv+Zw)\,d\Omega\,dt$$

$$+\frac{1}{2\Omega\tau}\iiint_{=o}^{t=\tau}(X_nu+Y_nv+Z_nw)\,dS\,dt \qquad (5)$$

Since if τ be sufficiently large, the function $\frac{d}{dt}(u^2+v^2+w^2)$ will have the same value at the beginning and end of the process if the motion be vibratory, for then τ will contain a large number of periods.

3. The analogy of theorem (5) to Clausius's[2] Virial Theorem is quite evident. According to the virial theorem, we have

$$-T=\tfrac{1}{2}\,\Sigma\Sigma\,xX+yY+zZ,$$

where T=kinetic energy of the number of particles within unit volume, (X, Y, Z)=force components on the particle which occupies the point (x, y, z).

4. A number of interesting applications are at once suggested.

Suppose the motion to be vibratory. Then if the body forces be nil, the average kinetic energy will be equivalent to the average potential energy if

(*i*) the surface tractions be nil, or constant, as in the case of the vibration of a supported rod, or plate with free ends,

(*ii*) the surface displacement be zero,

(*iii*) if part of the surface be under zero or constant stress and part under varying stress with no surface displacement (e.g. the case of a clamped rod, or string stretched between two points).

These theorems are of course well known, and can be deduced in other ways.

[1] Love's Elasticity, p. 83.

[2] *Vide* Jeans' Dynamical Theory of Gases, Second Edition, p. 141.