85. VERTICAL PROPAGATION OF ELECTROMAGNETIC WAVES IN THE IONOSPHERE

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SUMMARY

A connected discussion of the equations for the vertical propagation of e.m. waves in the ionosphere is given in standardised notation. It is shown that the electric field vector components E_x and E_y are coupled by polarisation terms, ρ_1, ρ_2 which are functions of G. M. latitude and height; and the propagation vectors, V and W, equal respectively to $(E_x+i\rho E_y)/\sqrt{1+\rho^2}$, for two values of ρ, are governed by two refractive indices q_0, q_e, and a coupling term $\dot{\phi}$; V and W may be identified with *o*- and *e*- waves respectively. The five quantities needed to define wave propagation completely are ρ_1, ρ_2, $\dot{\phi}$, q_0 and q_e; we have given a detailed discussion of the first three, and have omitted discussions about q_0 and q_e which are identical with those given by Appleton and have been discussed in detail by Booker (1935) and others. It is shown that the coupling term $\dot{\phi}$ can be neglected everywhere for *F*-layer propagation except very near the G.M. poles, while the *E*-layer propagation is more difficult to handle.

INTRODUCTION

Wave equations for the propagation of e.m. waves in the ionosphere had been given by Hartree (1929), Epstein (1930), Försterling (1931), Saha and Rai (1937), Rawer (1939), and Rydbeck (1940) and more recently by B. K. Banerjea (1947). Most of the older investigations were confined to the particular case of vertical propagation in the magnetic equator or poles. In recent years vertical propagation in any latitude has been tackled by Rydbeck (1941) and by Saha, Banerjea, Guha (1947), but a full discussion of the equations is still wanting. We discuss here the exact equations from a standpoint which is likely to throw light on the nature of modification of wave propagation at high geomagnetic latitudes.

§1. DEDUCTION OF EQUATIONS OF WAVE PROPAGATION

The deduction of the equations of propagation of e.m. waves in the ionosphere follows from the usual method of e.m. theory of light. We start with the Maxwell's equations:

$$\nabla\times\mathbf{H}=\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t};\quad \nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t}$$
$$\nabla.\mathbf{D}=0;\quad \nabla.\mathbf{H}=0;\quad \mathbf{D}=\mathbf{E}+\mathbf{P}=\mathbf{K}.\mathbf{E}. \tag{1.1}$$

It will be presently shown that the dielectric constant **K** is a tensor quantity. Here **P** is the polarisation vector due to the displacement of electrons by the electric field of the wave as modified by the presence of the earth's magnetic field. If **S** denotes the displacement of the electron under these conditions, we have

$$\mathbf{P}=4\pi N e\mathbf{S} \quad .. \quad .. \quad .. \tag{1.2}$$

The displacement **S** is given by the Lorentz equation

$$\frac{d^2\mathbf{S}}{dt^2}+\nu\frac{d\mathbf{S}}{dt}+\frac{e}{mc}\left[\mathbf{H}\times\frac{d\mathbf{S}}{dt}\right]=\frac{e}{m}.\mathbf{E} \quad .. \quad .. \tag{1.3}$$

Replacing **S** by **P** and since **E**, **S** $\propto e^{ipt}$ we can easily find out the solution of (1.3) in the form

$$\mathbf{P}=A.\triangle\mathbf{E}, \tag{1.2a}$$

where $A=r/\beta(\beta^2-\omega^2)$ and $\triangle$ is a tensor quantity given by the matrix

$$\triangle=\begin{pmatrix}\omega_x^2-\beta^2 & \omega_y\omega_x+i\beta\omega_z & \omega_z\omega_x-i\beta\omega_y\\ \omega_x\omega_y-i\beta\omega_z & \omega_y^2-\beta^2 & \omega_z\omega_y+i\beta\omega_x\\ \omega_x\omega_z+i\beta\omega_y & \omega_y\omega_z-i\beta\omega_x & \omega_z^2-\beta^2\end{pmatrix} \tag{1·4}$$

Here $\beta=1-i\nu/p$ where ν=collision-frequency, $r=p_0{}^2/p^2$ where $p_0{}^2=4\pi Ne^2/m$, N being the number of particles per c.c., $\omega=\mathbf{p}_h/p$ where $\mathbf{p}_h$=circular gyromagnetic frequency= $e\mathbf{H}/mc$, ω_x, ω_y, ω_z are components of ω.

From (1.1), (1.2) and (1.4) we can write the complex dielectric tensor in the form

$$\mathbf{K}=\begin{pmatrix}1-A(\beta^2-\omega_x^2) & A(\omega_x\omega_y+i\beta\omega_z) & A(\omega_z\omega_x-i\beta\omega_y)\\ A(\omega_x\omega_y-i\beta\omega_z) & 1-A(\beta^2-\omega_y^2) & A(\omega_z\omega_y+i\beta\omega_x)\\ A(\omega_x\omega_z+i\beta\omega_y) & A(\omega_z\omega_y-i\beta\omega_x) & 1-A(\beta^2-\omega_z^2)\end{pmatrix} \tag{1.4a}$$

So far the treatment has been quite general. Let us now consider the propagation of a plane wave along the vertical directions (axis of z) and the axis of y is chosen to be perpendicular to the magnetic meridian. Then putting $\omega_y=0$, $\mathbf{K}$ comes out as

$$\mathbf{K}=\begin{pmatrix} 1-A(\beta^2-\omega_x^2) & Ai\beta\omega_z & A\omega_x\omega_z \\ -Ai\beta\omega_z & 1-A\beta^2 & Ai\beta\omega_x \\ A\omega_x\omega_z & -Ai\beta\omega_x & 1-A(\beta^2-\omega_z^2) \end{pmatrix} \quad (1\cdot4b)$$

Introducing the condition $\nabla.\mathbf{D}=0$, i.e. $E_z+P_z=0$ we get from $(1\cdot4b)$

$$E_z=\frac{r\omega_z}{C'}(-\omega_z E_x+i\beta E_y)$$

where $$C'=\beta(\beta^2-\omega^2)-r(\beta^2-\omega^2) \quad (1\cdot5)$$

Taking the curl of both sides of the second equation in (1.1) and putting $\frac{\partial}{\partial t}=ip$ we get the wave equation for the electric vector in the form

$$\nabla\times\nabla\times\mathbf{E}-\frac{p^2}{c^2}\mathbf{D}=0 \quad (1\cdot6)$$

Breaking up this equation into components and putting $\frac{\partial}{\partial x}=\frac{\partial}{\partial y}=0$ we get $\frac{d^2E_x}{dz^2}+\frac{p^2}{c^2}D_x=0, \frac{d^2E_y}{dz^2}+\frac{p^2}{c^2}D_y=0$ (1.7)

From (1.6) and (1.7) we get after some simplification the following expressions for the Maxwell Displacement Vector

$$D_x=K_1E_x-iLE_y \qquad D_y=K_2E_y+iLE_x$$
$$K_1=1-r\frac{\beta^2-r\beta-\omega_x^2}{C'},\ K_2=1-\frac{r(\beta^2-r\beta)}{C'} \quad (1\cdot8)$$
$$L=r(r-\beta)\omega_z/C'$$

Introducing the symbol $u=zp/c$, we get the equations of propagation of the electric vector as

$$\frac{d^2E_x}{du^2}+K_1E_x-iLE_y=0;\ \frac{d^2E_y}{du^2}+K_2E_y+iLE_x=0 \quad (1\cdot9)$$

Equations in these forms do not help us much in the understanding of the phenomenon unless the coupling term L vanishes, or $K_1=K_2$. We have $L=0$ when $\omega_z=0$, i.e. at the magnetic equator (quasi-transverse case).

$K_1=K_2$ when $\omega_x=0$; this holds only for the magnetic poles (quasi-longitudinal case). These special cases are given in equations (1·17), (1·16).

For any latitude we try the following procedure. Multiply both sides of the second equation in (1·9) by some indeterminate quantity $i\rho$ and add the product to the first equation. Re-arranging the terms we get

$$\frac{d^2}{du^2}(E_x+i\rho E_y)+(K_1-\rho L)E_x+(K_2-L/\rho)i\rho E_y-2i$$
$$\frac{d\rho}{du}\frac{dE_y}{du}-i\frac{d^2\rho}{du^2}E_y=0 \quad (1\cdot10)$$

Since we are free to choose ρ, it is advantageous to do it in such a way that the coefficients of E_x and $i\rho E_y$ are equal, i.e. we put

$$K_1-\rho L=K_2-L/\rho=q^2 \quad (1.11)$$

ρ is therefore given by the roots of the equation

$$\rho^2-(K_1-K_2)\rho/L-1=0 \quad (1.12)$$

Now introducing the symbol

$$G=(K_1-K_2)/2L=\frac{\omega_x^2}{2\omega_z(r-\beta)} \quad (1.13)$$

the two roots of equation (1·11), which can be denoted by ρ_1 and ρ_2, can be written as:

$$\rho_1=G-\sqrt{1+G^2};\ \rho_2=G+\sqrt{1+G^2} \quad (1.14)$$

Now it can be easily shown that

$$C'=(\beta-r)(\beta+\rho_1\omega_z)(\beta+\rho_2\omega_z).$$

With the aid of this relation, it is easy to prove from (1.11) that q^2 has two values q_o^2 and q_e^2 given by

$$q_o^2=1-\frac{r}{\beta+\rho_1\omega_z},\ q_e^2=1-\frac{r}{\beta+\rho_2\omega_z} \quad (1.14a)$$

It is easy to see that these expressions are equivalent to Appleton expressions (for complex refractive index)[2]. (Vide the expression given by him in the Report on the Progress of Physics, Vol. 2, 1936.)

We may now turn to the equations (1.10). If $d\rho/du$ and $d^2\rho/du^2$ can be neglected, we have the following equations of propagation for the two waves

$$\left.\begin{aligned} O\text{-wave: } & \frac{d^2}{du^2}(E_x+i\rho_1E_y)+q_o^2(E_x+i\rho_1E_y)=0 \\ X\text{-wave: } & \frac{d^2}{du^2}(E_x+i\rho_2E_y)+q_e^2(E_x+i\rho_2E_y)=0 \end{aligned}\right\} \quad (1.15)$$

The terms we have neglected are

$$-2i.\frac{d\rho}{du}.\frac{dE_y}{du}-i\frac{d^2\rho}{du^2}E_y.$$

Now we have

$$\frac{d\rho}{du}=\frac{d}{du}(G\mp\sqrt{1+G^2})\simeq\frac{dr}{dz}\simeq\frac{dN}{dz}$$

and can be neglected only when $\frac{dN}{dz}=0$ or small. Equations (1.15) therefore hold when we can neglect dN/dz and d^2N/dz^2, i.e. for a medium for which N varies very slowly.

The equations (1.15) take simplified forms at the geomagnetic equator and the poles. For the GM-equator we have, since $G=\infty$,

$$\rho_1=G-\sqrt{1+G^2}=0,\ \omega_z\rho_2=2G\omega_z=\frac{\omega^2}{r-\beta}$$

Hence we have

$$q_o^2=1-\frac{r}{\beta},\ q_e^2=1-\frac{r}{\beta-\frac{\omega^2}{\beta-r}}$$

and the equations (1.15) have the form

$$\left.\begin{aligned}&(O\text{-wave}):\ \frac{d^2E_x}{du^2}+\left(1-\frac{r}{\beta}\right)E_x=0\\&(X\text{-wave}):\ \frac{d^2E_y}{du^2}+\left(1-\frac{r}{\beta-\frac{\omega^2}{\beta-r}}\right)E_y=0\end{aligned}\right\}\quad(1.16)$$

For the poles, we have $G=0$, $\rho_2=1$, $\rho_1=-1$, and $\omega_z=-\omega$ at the north pole, and $=\omega$ at the south pole. Hence we have the equations

$$\left.\begin{aligned}&(O\text{-wave}):\frac{d^2}{du^2}(E_x-iE_y)+\left(1-\frac{r}{\beta\pm\omega}\right)(E_x-iE_y)=0\\&(X\text{-wave}):\frac{d^2}{du^2}(E_x+iE_y)+\left(1-\frac{r}{\beta\mp\omega}\right)(E_x+iE_y)=0\end{aligned}\right\}\quad(1.17)$$

The upper sign refers to the N-magnetic pole, the lower to the S-magnetic pole.

More rigorous equations of propagation can be deduced by rotating the axes through a complex angle ϕ. We start from (1.9) and put

$$\begin{pmatrix}E_x\\iE_y\end{pmatrix}=\begin{pmatrix}\cos\phi&-\sin\phi\\\sin\phi&\cos\phi\end{pmatrix}\begin{pmatrix}V\\W\end{pmatrix}=S\begin{pmatrix}V\\W\end{pmatrix}$$

where S denotes the matrix given above. The inverse matrix

$$S^{-1}=\begin{pmatrix}\cos\phi&\sin\phi\\-\sin\phi&\cos\phi\end{pmatrix}\text{ and we have }\begin{pmatrix}V\\W\end{pmatrix}=S^{-1}\begin{pmatrix}E_x\\iE_y\end{pmatrix}$$

Then it can be easily shown that if by $\dot{E}_x$ we denote $\frac{dE_x}{du}$, etc.

$$\begin{pmatrix}\dot{E}_x\\i\dot{E}_y\end{pmatrix}=S\begin{pmatrix}V'\\W'\end{pmatrix},\ \begin{pmatrix}\ddot{E}_x\\i\ddot{E}_y\end{pmatrix}=S\begin{pmatrix}V''\\W''\end{pmatrix}$$

where

$$\dot{V}=\frac{dV}{du},\quad \ddot{V}=\frac{d^2V}{du^2},\text{ etc;}$$

$$V'=\dot{V}-\dot{\phi}W,\quad W'=\dot{W}+\dot{\phi}V;$$

$$V''=\frac{dV'}{du}-\dot{\phi}W'=\ddot{V}-2\dot{\phi}\dot{W}-\ddot{\phi}W-\dot{\phi}^2V;$$

$$W''=\frac{dW'}{du}+\dot{\phi}V'=\ddot{W}+2\dot{\phi}\dot{V}+\ddot{\phi}V-\dot{\phi}^2W.$$

It is easy to see that V', W' are the moving co-ordinate derivatives of V and W. Hence the fundamental equations (1.9) can be written as

$$(V''+K_1V-LW)\cos\phi-(W''+K_1W+LV)\sin\phi=0,\quad(1.18)$$
$$(V''+K_2V+LW)\sin\phi-(W''+K_2W-LV)\cos\phi=0.$$

From these equations, it is easy to deduce the following equations:

$$V''+\{K_1\cos^2\phi+K_2\sin^2\phi-2L\sin\phi.\cos\phi\}V$$
$$-\{(K_1-K_2)\sin\phi.\cos\phi+L(\cos^2\phi-\sin^2\phi)\}W=0\quad(1.19a)$$
$$W''+\{K_1\sin^2\phi+K_2\cos^2\phi+2L\sin\phi\cos\phi\}W$$
$$-\{(K_1-K_2)\sin\phi.\cos\phi+L(\cos^2\phi-\sin^2\phi)\}V=0\quad(1.19b)$$

We observe that the coefficient of cross-terms, i.e. of W in (1.19a) and of V in (1·19b) have the same value. This may be made to vanish by appropriate choice of ϕ,

i.e. putting $(K_1-K_2)/2L=G=-\cot 2\phi$ (1·20)

If we put $\tan\phi=\tau$, the above equation is equivalent to $\tau^2-2\tau G-1=0$, or $\tau=\tan\phi=G\pm\sqrt{1+G^2}=\rho_1,\ \rho_2$.
As before, we denote by ρ_1 the quantity $G-\sqrt{1+G^2}$. Then the other value of

$$\tau=-\frac{1}{\rho_1}=\rho_2=G+\sqrt{1+G^2}.$$

It can be easily shown that the coefficients of V and W in (1.19a) and (1.19b) viz.,

$$K_1\cos^2\phi+K_2\sin^2\phi-2L\sin\phi.\cos\phi=q_o^2$$
$$K_1\sin^2\phi+K_2\cos^2\phi+2L\sin\phi.\cos\phi=q_e^2$$

We have therefore the final equations:

$$V''+q_o^2V=0;\quad W''+q_e^2W=0$$

Now

$$V=E_x\cos\phi+iE_y\sin\phi=\frac{E_x+i\rho_1E_y}{\sqrt{1+\rho_1^2}}$$

$$W=-E_x.\sin\phi+iE_y\cos\phi=\frac{E_x+i\rho_2E_y}{\sqrt{1+\rho_2^2}}\quad(1.21)$$

Hence the equations written in full are:—

$$\left.\begin{aligned}&(O\text{-wave})\quad\frac{d^2V}{du^2}+(q_o^2-\dot{\phi}^2)V=2\dot{\phi}\dot{W}+\ddot{\phi}W\\&(X\text{-wave})\quad\frac{d^2W}{du^2}+(q_e^2-\dot{\phi}^2)W=-2\dot{\phi}\dot{V}-\ddot{\phi}V\end{aligned}\right\}^{1}\quad(1.22)$$

If the coupling term $\dot{\phi}=\frac{d}{du}\tan^{-1}\rho=\frac{d\rho/du}{1+\rho^2}$ can be put equal to zero, then equations (1.22) reduce to the equations (1.15). The coupling term is discussed in detail in the next section.

The advisability of rotating the axes through an angle ϕ is easily suggested from analogy of the present case to that of crystal optics. We write the equation (1.7) in the form

$$\frac{d^2\mathbf{E}}{du^2}+\mathbf{D}=0$$

[1] This type of equations was given by Rydbeck in 1950, but the signs of the right hand side of the two equations were given as same. This is clearly a mistake, the signs ought to be opposite.

where the vector **E** has E_x and iE_y as the components and the vector **D** is given by

$$\begin{pmatrix} D_x \\ iD_y \end{pmatrix} = \begin{pmatrix} K_1 & -L \\ -L & K_2 \end{pmatrix} \begin{pmatrix} E_x \\ iE_y \end{pmatrix}$$

Since the tensor connecting **D** and **E** is a symmetrical tensor we write the equation of the tensor-ellipsoid as

$$\mathbf{D.E}=K_1E_x^2-2LE_x.iE_y+K_2(iE_y)^2$$

from which we find out the principal axes, by the customary method of removing the cross term. We put

$$E_x=V\cos\phi-W\sin\phi,\quad iE_y=V\sin\phi+W\cos\phi.$$

This is equivalent to our transformation. The rest is identical.

The tensor-ellipsoid now takes the form

$$\mathbf{D.E}=q_o^2V^2+q_e^2W=f.$$

ϕ is the angle between the major axis of the ellipse and the magnetic meridian.

If $\frac{dN}{dz}=0$ it retains a constant value. (See discussion in the next section.) We observe from (1·22) that the two modes of propagation are characterised by the phase velocities c/q_o, c/q_e and their respective polarisations are given by $i\rho_1$ and $i\rho_2$ (*vide infra.*)

It is not easy to deduce simple equations for the magnetic vectors. They are of the fourth degree. But some simplified relations can be deduced from the fundamental Maxwell equations. From the relation (1.1) we have

$$\frac{dH_x}{du}=-LE_x+iK_2E_y,\quad i\frac{dH_y}{du}=K_1E_x-L.iE_y$$

or expressing E_x and iE_y in terms of V and W, we have

$$\begin{pmatrix} \dot H_x \\ i\dot H_y \end{pmatrix} = \begin{pmatrix} K_2\sin\phi-L\cos\phi & K_2\cos\phi+L\sin\phi \\ K_1\cos\phi-L\sin\phi & -K_1\sin\phi-L\cos\phi \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

or

$$\left.\begin{aligned} V&=(\dot H_x\sin\phi+i\dot H_y\cos\phi)/q_o^2=E_x\cos\phi+iE_y\sin\phi \\ W&=(\dot H_x\cos\phi+i\dot H_y\sin\phi)/q_e^2=-E_x\sin\phi+iE_y\cos\phi \end{aligned}\right\}\quad (1.23)$$

From these expressions, which are exact, it is not possible to deduce any simple relation between the magnetic and electric vectors at any point. Only at the ground ($r=0$) where

$$q_o^2=q_e^2=1,\text{ we have } E_x=i\dot H_y,\ iE_y=\dot H_x \quad (1.24)$$

but these relations are not applicable for any other point.

§2. Polarisation and Coupling.

Expressions (1·22) represent the most rigorous equations for vertical propagation of e.m. waves in the ionosphere. The electric vectors E_x, E_y are coupled by the quantities ρ_1 and ρ_2 forming two new vectors V and W given by (1.21) and equations for V and W are coupled by the factor $\dot\phi$. As shown earlier V may be identified with the electric vector for the *o*-mode of propagation, with the corresponding quantity W for the *e*-mode.

The general form of solution of (1.22) can be written in the forms:

$$\left.\begin{aligned} V&=(E_x+i\rho_1E_y)/\sqrt{1+\rho_1^2}\simeq e^{ipt}.\,f_V(z) \\ \text{and}\quad W&=(E_x+i\rho_2E_y)/\sqrt{1+\rho_2^2}\simeq e^{ipt}.\,f_W(z) \end{aligned}\right\}\quad (2.1)$$

or since ρ_1 and ρ_2 are both functions of z, we can write:

$$E_x+i\rho_1E_y\simeq e^{ipt}.\psi_V(z),\quad E_x+i\rho_2E_y\simeq e^{ipt}.\psi_W(z)$$

where both ψ_V and ψ_W are complex functions of the real variable z.

Let us now put

$$i\rho_1=Re^{i\alpha}\text{ and consequently } i\rho_2=R^{-1}e^{-i\alpha} \quad (2.1a)$$

Therefore we have

$$\left.\begin{aligned} &E_x.e^{-ipt}+RE_ye^{-i(pt-\alpha)}=\psi_1(z) \\ &E_x\cos pt+R.E_y\cos(pt-\alpha)=Re\,|\,\psi_1(z)\,|=A_V \\ &E_x\sin pt+RE_y\sin(pt-\alpha)=Im\,|\,\psi_1(z)\,|=B_V \end{aligned}\right\}\quad (2.2)$$

where A and B are functions of z.
Hence we have for the V-wave:

$$E_x^V=\frac{C_V}{\sin\alpha}\sin(pt-\alpha+\pi-\theta_V),\ E_y^V=\frac{C_V}{R\sin\alpha}\sin(pt-\theta_V) \quad (2.3)$$

where $C_V=\sqrt{A_V^2+B_V^2},\quad \theta_V=\tan^{-1}(B_V/A_V).$

Similarly for the W-wave, we get

$$E_x^W=\frac{RC_W}{\sin\alpha}\sin(pt-\alpha-\theta_W),\quad E_y^W=\frac{C_W}{\sin\alpha}\sin(pt+\pi-\theta_W)$$

where

$$C_W=\sqrt{A_W^2+B_W^2},\quad \theta_W=\tan^{-1}(B_W/A_W) \quad (2.4)$$

The phase-angle α and the amplitude ratio R are in general functions of δ, ω, r and θ, and we discuss presently how to obtain analytical expressions for R and α.

Magnetic Damping Factor: We denote the magnetic condition of the locality by the angle of propagation θ, which is the angle between the upward vertical direction and the positive direction of the magnetic lines of force. Therefore

$$\omega_z=-|\,\omega\,|\cos\theta,\quad \omega_x=-|\,w\,|\sin\theta \quad (2.5)$$

because $\omega=\frac{e\mathbf{H}}{mcp}$, and $e=-|\,e\,|$ for electrons;

put $$\nu_c=p\omega_X^2/2\omega_z=-\frac{|\,p_h\,|}{2\cos\theta}.\sin^2\theta. \quad (2.6)$$

ν_c is positive in the N.H. and negative in S.H. $|\,\nu_c\,|$ varies from 0 at the poles to ∞ at the equator, and its value for a number of stations in the N.H. is shown in row 6 of Table III. Let us denote ν_c/p by δ_c which may be termed the magnetic damping ratio. Then introducing the quantities ξ and η defined by

$$\xi=\frac{\delta}{\delta_c},\quad \eta=\frac{1-r}{\delta_c},\quad \tan\gamma=\frac{\delta}{1-r} \tag{2.7}$$

we have $G=\frac{\delta_c}{r-\beta}=\frac{1}{-\eta+i\xi}=-\left(\frac{\eta}{\eta^2+i\xi}+i\frac{\xi}{\eta^2+\xi^2}\right)$ (2.8)

We shall now show that it is convenient to represent R, α and other quantities on a ξ, η-plane. Let us first discuss the possible range of values of ξ and η.

THE ξ, η-PLANE

We have $\xi=\nu/\nu_c$; hence ξ is positive in the N.H., and negative in the S.H. $|\xi|$ varies from ∞ at the poles to 0 at the G.M.E. Large values of ξ denote large damping, small values small damping. $\xi=0$ indicates no damping.

As $|\delta_c|$ varies from 0 at the pole to ∞ at the **G.M.E.,** the particular values of ξ which refer to E or F-regions at any locality will have to be found for every locality separately.

Let us now trace the values $\eta=\frac{1-r}{\delta_c}$. In the N.H., since δ_c is positive, we have on the ground $r=0$, $\eta=1/\delta_c$. This is the maximum value of η below $r=1$. As we take higher points in the ionosphere, $r=\frac{4\pi Nc^2}{mp^2}$ becomes appreciable, the value of η decreases and it becomes zero at $r=1$, after which for $r>1$, η is negative. Therefore the line $r=1$, which corresponds to $\eta=0$, divides the $\xi\eta$-plane in two parts.

Let us now try to find out general analytical expressions for R and α at any height, i.e. for any value of r and δ, as functions of ξ and η. We proceed from the expressions

$$\left.\begin{aligned} Re^{i\alpha}&=i\rho_1=iG\{1-\sqrt{1+1/G^2}\}\\ R^{-1}e^{-i\alpha}&=i\rho_2=iG\{1+\sqrt{1+1/G^2}\}\end{aligned}\right\} \tag{2.9}$$

Now using (2.6) and (2.7), we have

$$(R+R^{-1})\cos\alpha+i(R-R^{-1})\sin\alpha=2iG$$

from which we deduce the two relations

$$(R+R^{-1})\cos\alpha=\frac{2\xi}{\eta^2+\xi^2},\ (R-R^{-1})\sin\alpha=-\frac{2\eta}{\eta^2+\xi^2} \tag{2.10}$$

From (2.9), we deduce the important relation

$$\cos 2\alpha=\frac{1}{\lambda^2}-\sqrt{1+\frac{2}{\lambda^2}\cos 2\gamma+\frac{1}{\lambda^4}} \tag{2.11}$$

where $\lambda^2=\xi^2+\eta^2$

The sign before the radical is ± 1. We have retained only the negative sign after comparison with the limiting case of a friction-free atmosphere. The following expressions for R can be easily verified

$$\left.\begin{aligned} R^2&=\frac{\xi\tan\alpha-\eta}{\xi\tan\alpha+\eta}\\ R&=\frac{1}{\lambda^2}\left\{\frac{\xi}{\cos\alpha}-\frac{\eta}{\sin\alpha}\right\}\\ R^{-1}&=\frac{1}{\lambda^2}\left\{\frac{\xi}{\cos\alpha}+\frac{\eta}{\sin\alpha}\right\}\end{aligned}\right\} \tag{2.12}$$

There are some inherent ambiguities in these expressions for R and α which must be removed. This can be done if we follow the course of the complex quantities ρ_1 and ρ_2 in the complex plane and always confine ourselves to one particular branch in discussing the nature of ρ_1, ρ_2.

For this purpose we start with expressions (2.9).

Now

$$1+\frac{1}{G^2}=\{1-\xi^2+\eta^2\}-2i\xi\eta \tag{2.13}$$

when $r=0$, $Im\left\{1+\frac{1}{G^2}\right\}=-2\xi\eta$ and it is definitely negative. Hence, if we plot $1+\frac{1}{G^2}$ in the complex plane, the point representing it must be in the third or the fourth quadrant. Therefore, the point $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ which is double-valued, must have one of its values in the fourth quadrant, and therefore, the other will be in the second quadrant.

We now choose $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ to be given by the point in the fourth quadrant for $r=0$.

Now let r increase; when $r=1$, $Im\left\{1+\frac{1}{G^2}\right\}=0$ and as (2.11) shows

$Re\left\{1+\frac{1}{G^2}\right\}>0$ if $\xi<1$, and <0 if $\xi>1$.

Hence, at $r=0$, and for $\xi<1$, $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ is either real positive or real negative. But the last alternative is ruled out as we have $\left\{1+\frac{1}{G^2}\right\}^{\frac{1}{2}}$ in the fourth quadrant. Hence $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ is real positive for $r=1$, $\xi<1$. Similarly for $\xi<1$, $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ is negative imaginary. For $r>1$, and $\xi<1$, $Im\left\{1+\frac{1}{G^2}\right\}$ is >0, and $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ is in the first quadrant, while for $\xi>1$, $\left(1+\frac{1}{G^2}\right)^{\frac{1}{2}}$ is in the third quadrant, i.e., the real part is negative.

Thus $Re\left\{1+\frac{1}{G^2}\right\}^{\frac{1}{2}}$ is always positive except when $\xi>1$ and $r>1$. In this case $Re\left\{1+\frac{1}{G^2}\right\}^{\frac{1}{2}}$ is negative.

Now from (2.9), we have

$$\frac{|\rho_1|}{|\rho_2|}=R^2=\left(\frac{1-2A\cos\beta+A^2}{1+2A\cos\beta+A^2}\right)^{\frac{1}{2}}$$

where $A\cos\beta=Re\left\{1+\frac{1}{G^2}\right\}^{\frac{1}{2}}$

Hence, in general $R^2\leqslant 1$, except for the quadrant $\xi>1$, $r>1$, in which case, since $Re\left\{1+\frac{1}{G^2}\right\}^{\frac{1}{2}}<0$, $R^2>1$.

Putting $\tan\gamma=t$, we have from (2.10)

$$\cos\alpha=\frac{1}{\xi}\cdot\frac{2t^2}{1+t^2}\cdot\frac{R}{R^2+1}$$

$$\sin\alpha=\frac{1}{\xi}\cdot\frac{2t}{1+t^2}\cdot\frac{R}{1-R^2}$$

Let us take N.H. Since ξ is positive, we have $\cos\alpha>0$ and $\sin\alpha>0$ for $r<1$; and for $r>1$ and $\eta<1$, $\sin\alpha$ is negative; for $r>1$ and $\xi>1$, $\sin\alpha>0$.

For ready reference we have given in Table I the signs and ranges of R and α for the different regions in the $\xi\eta$-plane.

Before taking the general case, let us consider how R and α vary along the central line $\eta=0$. We have now $G=-i/\xi$, hence, we have

$$Re^{i\alpha}=\frac{1}{\xi}\left\{1-\sqrt{1-\xi^2}\right\}$$

$$\left.\begin{array}{ll}\text{Now if} & \xi<1,\ \alpha=0,\ R=\frac{1}{\xi}-\left(\frac{1}{\xi^2}-1\right)^{\frac{1}{2}}\\ \text{but if} & \xi>1,\ Re^{i\alpha}=\frac{1}{\xi}\left\{1\pm i\left(1-\frac{1}{\xi^2}\right)^{\frac{1}{2}}\right\}\end{array}\right\}\quad(2.14)$$

Hence $R\cos\alpha=1/\xi$, $R\sin\alpha=\pm(1-1/\xi^2)^{\frac{1}{2}}$
$R=1$, $\cos\alpha=1/\xi$.

This shows that the line $\xi=1$, divides the $\xi\eta$-plane in two regions. On the left-hand side, we have $\alpha=0$ all along the abscissa, up to $\xi=1$; on the right-hand side $\alpha=\cos^{-1}1/\xi$ and, therefore, varies from 0 at $\xi=1$ to $\pi/2$ at $\xi=\infty$. The value of R on the abscissa is $=1$, if $\xi>1$ but for $\xi<1$ it is given by (2.14).

THE α=CONST. CURVES:

For the N.H. ξ is always positive. The lines $\xi=1$, and $\eta=0$ (figs. 1 and 2) divide the plane into four sections (I, II, III, IV) as shown above. We can find out the value of α with the aid of (2.11), and for its sign, we have to look to Table II. (2.11) can also be written as:

$$\frac{(\xi/\lambda^2)^2}{\cos^2\alpha}-\frac{(\eta/\lambda^2)^2}{\sin^2\alpha}=1 \qquad (2.15)$$

or

$$\frac{\lambda^2}{2}\sin^2 2\alpha+\cos 2\gamma+\cos 2\alpha=0 \qquad (2.15a)$$

When we wish to draw curve α=const., we can do so by using (2.11 or 2.15a). A few curves are given in fig. (1, 2). It is seen that for $\alpha>0$, these curves all cut the abscissa at right angles at $\xi=\sec\alpha$. The point (0, 0,) through which the curve appears to pass is to be excluded,

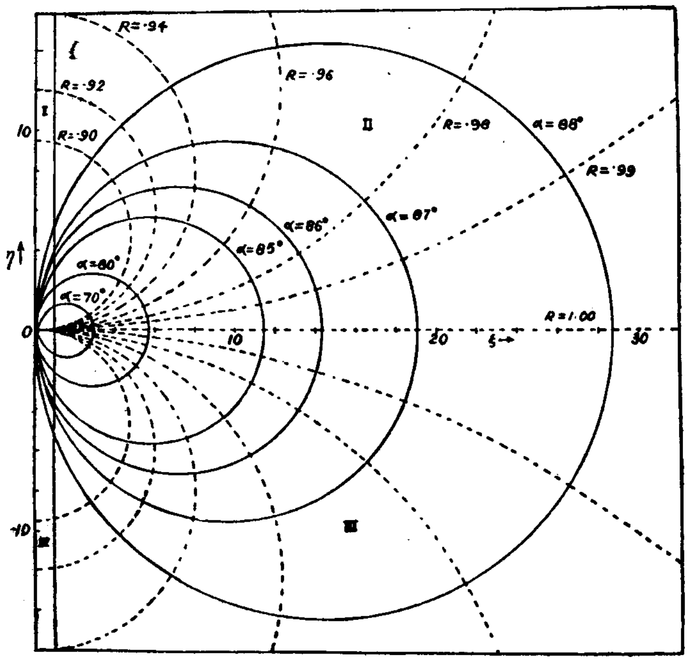

Fig. 1

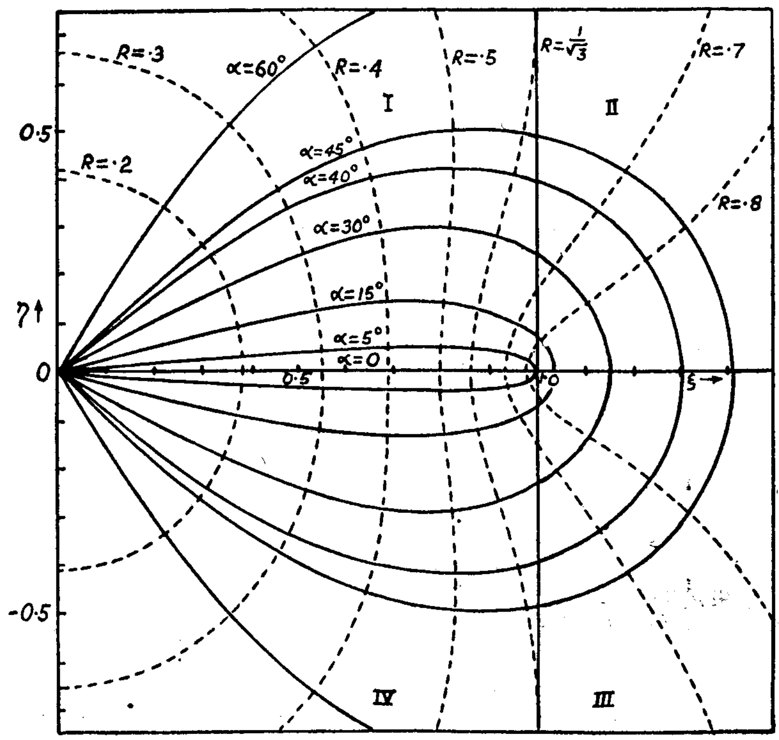

Fig. 2

but the curves approach this point at the angle $\pi/2-\gamma$ to the abscissa.

The sign of α is positive in sections I, II, III, but is negative in section IV. We have $\alpha=0$ for the abscissa up to $\xi=1$; on the ordinate $\alpha=\pi/2$ above the abscissa and $=-\pi/2$ below.

Let us see how to draw the curves R=const. It can be shown from (2.12) that

$$\left(\frac{\xi}{\lambda^2}\right)^2 \Big| \left(\frac{2R}{1+R^2}\right)^2+\left(\frac{\eta}{\lambda^2}\right)^2 \Big| \left(\frac{2R}{1-R^2}\right)^2=1. \quad (2.16)$$

Now let us put $R=\tan\psi$; then $\sin 2\psi=\frac{2R}{1+R^2}$, $\tan 2\psi=\frac{2R}{1-R^2}$; we can rewrite in the form

$$\lambda^2=\sin^2 2\psi \,\{1+\tan^2 2\psi \cos^2\gamma\} \quad (2.16a)$$

This gives us the polar equation in the $\xi\eta$-plane for R=const. curves; when $R<<1$ we have $\sin 2\psi=2R<<1$ and (2.16a) reduces to $\lambda^2=(2R)^2$, i.e., the R=const. curves are circles. But as R becomes larger, we have curves as shown in figs. (1, 2). We observe that for $R<1$, these curves cut the abscissa ($\gamma=\pi/2$) at $\xi=\sin 2\psi=\frac{2R}{1+R^2}$, and the ordinate ($\gamma=0$) at $\eta=\tan 2\psi=\frac{2R}{1-R^2}$. The curves crowd as we approach the point $\xi=1$.

So far the R=const. curves are confined to sections I and IV. Let us now take sections II and III. From the expression

$$R^2=(\xi \tan\alpha-\eta)/(\xi\tan\alpha+\eta)$$

We have $R\leqslant 1$ in section II, but in Section III we have

$$R^2=\frac{\xi\tan\alpha+|\eta|}{\xi\tan\alpha-|\eta|}>1$$

i.e., at the mirror points (ξ,η), $(\xi,-\eta)$, the values of R are reciprocal to each other. In Section II, $R<1$, and in section III, $R>1$.

The R=constant curves for graded values of R are shown in figs. (1 and 2). In I, II and IV, $R<1$, while in III R should be read as $1/R$, so that $R>1$ in this section. It is to be noticed that for $R<\frac{1}{\sqrt{3}}$ the curves are confined to the left-hand side of $\xi=1$. The curve $R=\frac{1}{\sqrt{3}}$ touches the line $\xi=1$, and the curves with $R>\frac{1}{\sqrt{3}}$ cut $\xi=1$ at two points. For values of $R\simeq 1$, the R=constant curves are nearly circles with centres at $\left(0, \pm\frac{R}{1-R^2}\right)$ and radius $\frac{R}{1-R^2}$ except very near the point $\xi=0$, $\eta=0$. The family of R=constant curves cuts orthogonally the family of α=constant curves.

Scott (1950) has drawn curves similar to those given in figures (1 and 2).

Let us now study the polarisation of the incident and reflected waves in the light of the above discussion. The following table gives the phases for the general and some particular cases, of the eight field quantities E and H for both V and W-waves.

The figures in column 2 are obtained from (2.3), (2.4). For others see *infra.*

We have at present no means for determining θ_V and θ_W. For the ground, however, since $r=0$, $\delta=0$ and $\alpha=\pi/2$ for the N.H. we can easily write out the phase differences. They are given in column 3.

We have, therefore, for the ground

$$E_x^V=C_V\cos pt,\ E_y^V=\frac{C_V}{R}\sin pt,\ E_x^W=C_W\cos pt,$$

TABLE I

	Phase Angles	Phase Difference $r=0$, $\nu=0$	Phase Difference at $\eta=0$		Phase Difference	
			$\lvert\xi\rvert>1$, $\xi>0$	$\lvert\xi\rvert<1$, $\xi>0$	$\lvert\xi\rvert>1$, $\xi<0$	$\lvert\xi\rvert<1$, $\xi<0$
E_x^V	$\pi-\alpha-\theta_V$	$\pi/2$	$\pi-\sec^{-1}\lvert\xi\rvert$	π	$\sec^{-1}\lvert\xi\rvert$	0
E_y^V	$-\theta_V$	0	0	0	0	0
E_x^W	$\alpha-\theta_W$	$\pi/2$	$\sec^{-1}\lvert\xi\rvert$	0		π
E_y^W	$\pi-\theta_W$	π	π	π	$\pi+\sec^{-1}\lvert\xi\rvert$	π
H_x^V		0			π	
H_y^V		$-\pi/2$				
H_x^W		π				
H_y^W		$+\pi/2$				

$E_y^W = -RC_W \sin pt,$

$H_x^V = \frac{C_V}{R} \sin pt,\ H_y^V = -C_V \cos pt,\ H_x^W = -RC_W \sin pt,$

$H_y^W = -C_W \cos pt.$

The value of R can be obtained directly from (2.12) or from the condition $\eta = 2R/1(-R^2)$. We have generally for any point on $\xi=0$ (no damping),

$$R = \sqrt{1+{}^1/\eta^2} - \frac{1}{\eta}$$

and for $r=0$, we have

$$R = \sqrt{1+\delta_c^2} - \delta_c.$$

The polarisation-ellipses described by E and H for V and W-waves and their senses of rotation are shown in the figures given below for an assumed value of $R = \cdot 6$.

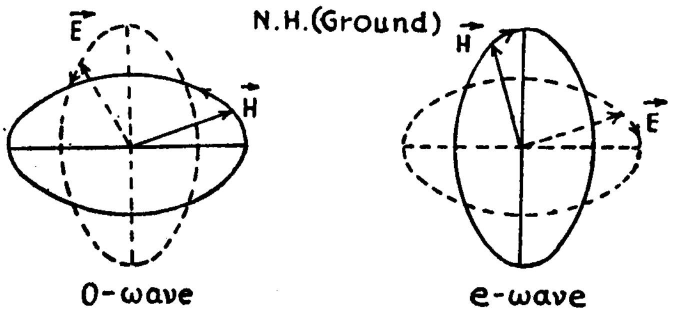

Fig. 3

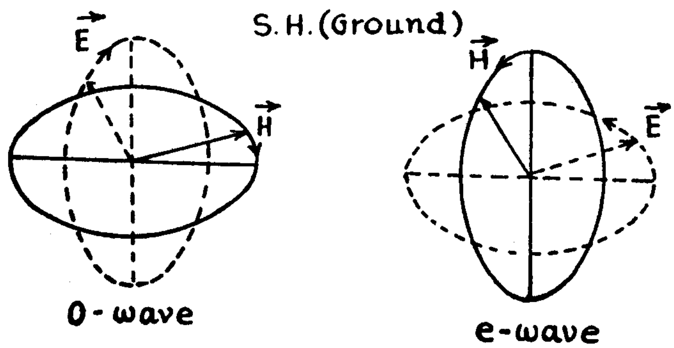

Fig. 4

It is instructive to see how the polarisation varies with height. For $r=1$ we can easily deduce the values of the relative phase angles for E with the aid of the above relations which are given in columns 4 and 5 of Table I. We have not given phase-angles for H, as relation (1.24) then no longer holds. We have now for $\xi=\infty$, $r=1$.

$E_x^V = C_V \cos pt,\ E_y^V = C_V \sin pt,\ E_x^W = C_W \cos pt,$

$E_y^W = -C_W \sin pt.$

These are circles as shown in fig. 5(e).

For $\xi > 1$, $r=1$, putting $\alpha = \cos^{-1}\frac{1}{\xi}$

$$E_x^V = -\frac{C_V}{\sqrt{1-1/\xi^2}} \sin(pt-\alpha),\quad E_y^V = \frac{C_V}{\sqrt{1-1/\xi^2}} \sin pt;$$

$$E_x^W = \frac{C_W}{\sqrt{1-1/\xi^2}} \sin(pt+\alpha),\quad E_y^W = -\frac{C_W}{\sqrt{1-1/\xi^2}} \sin pt.$$

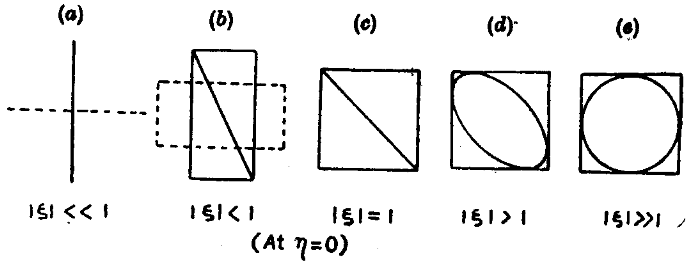

Fig. 5

These are ellipses, circumscribed within a square as shown in fig. 5(d), the contact points being given by (2.17a) and the ratio of axes are $\sqrt{(\xi-1)/(\xi+1)}$, the angle of tilt being $\pi/4$. For $\xi=1$, $r=1$,

$$E_x^V \simeq -\sin pt,\ E_y^V \simeq \sin pt,\ E_x^W \simeq \sin pt,\ E_y^W \simeq -\sin pt$$

The ellipse reduces to a straight line coincident with the diagonal of the square. For $\xi < 1$, $r=1$.

$$E_x^V = -\sin pt,\ E_y^V = R^{-1}.\sin pt,\ E_x^W \simeq \sin pt,\ E_y^W = R \sin pt$$

This is shown in fig. 5(c).

For $\xi > 1$, the curves are lines forming diagonals of two oblongs as shown in fig. 5(b), the ratio of the sides of the oblongs being given by R or R^{-1}.

When $\xi=0$, $r=1$, we have $R=0$, $R^{-1}=\infty$ and the polarisation ellipses reduce to two straight lines parallel to the magnetic axis and perpendicular to it (Fig. 5(a).)

Figures of polarisation ellipses for $r=1$ have been given by Booker (1934) for the magnetic vector. It will be seen that the diagonal lines in our figures are in the first and third quadrants. This is because we have drawn polarisation ellipses for E, while Booker has drawn for H.

In general, omitting θ_V and θ_W we have for any point of the ionosphere where R and α have the values given in Table II

$$E_x^V = -\frac{C_V}{\sin\alpha}.\sin(pt-\alpha),\quad E_y^V = \frac{C_V}{R\sin\alpha}\sin pt,$$

$$E_x^W = \frac{C_W}{\sin\alpha}\sin(pt+\alpha),\quad E_y^W = -\frac{RC_W}{\sin\alpha}\sin pt.$$

Now eliminating pt we deduce the following equations for the ellipses described by E_x and E_y:

$$\left.\begin{aligned}(E_x^V)^2 + 2R\cos\alpha\,.\,E_x^V E_y^V + R^2(E_y)^2 = C_V^2\\ (E_x^W)^2 + 2R^{-1}\cos\alpha\, E_x^W E_y^W + R^{-2}(E_y)^2 = C_W^2\end{aligned}\right\} \quad (2.17)$$

TABLE II

$\xi=\frac{\delta}{\delta c}$	$\eta=\frac{1-r}{\delta c}$	$ip=Re^{i\alpha}$		Polarisation of the electric vector	
		R	α	Nature of polarisation	Sense of rotation
$\vert\xi\vert=0,\ \delta=0$	$\eta>0$	$\frac{\sqrt{1+\eta^2}-1}{\vert\eta\vert}$	$+\frac{\pi}{2}$	Ellipse; ratio of axes$=R$; Tiltangle$=0$ (*o*-wave and *e*-wave)	Anticlockwise for *o*-wave and Clockwise for *e*-wave
	$\eta<0$	Do.	$-\frac{\pi}{2}$	Do.	Clockwise for *o*-wave and anti-clockwise for *e*-wave
$\vert\xi\vert<1$	$\eta=0$, $(r=1)$	$\frac{1-\sqrt{1-\xi^2}}{\vert\xi\vert}$	0 for $\xi>0$, π for $\xi<0$	Linear, tilt-angle $\tan^{-1}(-R)$ and $\tan^{-1}\left(\frac{1}{R}\right)$ for two waves	
$\vert\xi\vert=1$	$\eta=0$, $(r=1)$	1	Do.	Linear, tilt-angle$-\pi/4$ and $+\pi/4$.	
$\vert\xi\vert>1$	$\eta=0$, $(r=1)$	1	$\mathrm{Sec}^{-1}\vert\xi\vert$, 1st Quadrant for $\xi>0$, 3rd for $\xi<0$	Ellipse, ratio of axes $=\sqrt{\frac{\vert\xi\vert-1}{\vert\xi\vert+1}}$ Tilt-angle $-\pi/4$ and $+\pi/4$	Anticlockwise for *o*-wave and Clockwise for *e*-wave
	$\eta>0,\ \vert\xi\vert<1$	$R\leqslant1$	$0\leqslant\alpha\leqslant\pi/2$	Ellipse, defined by eqn. (2.17)	Anticlockwise for *o*-wave and Clockwise for *e*-wave
$\xi>0$, (N. H.)	$\vert\xi\vert>1$	$R\leqslant1$	$0\leqslant\alpha\leqslant\pi/2$	Do.	Do.
	$\vert\xi\vert<1$	$R\leqslant1$	$-\pi/2\leqslant\alpha\leqslant0$	Do.	Clockwise for *o*-wave and Anticlockwise for *e*-wave
	$\pi<0$, $\vert\xi\vert>1$	$R\geqslant1$	$0\leqslant\alpha\leqslant\pi/2$	Do.	Do.
$\xi<0$, (S. H.)	$\vert\xi\vert<1$	$R\leqslant1$	$\pi/2\leqslant\alpha\leqslant\pi$	Do.	Do.
	$\pi>0$, $\vert\xi\vert>1$	$R\geqslant1$	$\pi\leqslant\alpha\leqslant3\pi/2$	Do.	Do.
	$\vert\xi\vert<1$	$R\leqslant1$	$\pi\leqslant\alpha\leqslant3\pi/2$	Do.	Anticlockwise for *o*-wave and Clockwise for *e*-wave
	$\eta<0$, $\vert\xi\vert>1$	$R\leqslant1$	$\pi\leqslant\alpha\leqslant3\pi/2$	Do.	
$\xi\vert=0,\ \delta c=\infty,\ \delta\neq0$ Equator)	$\eta>0$, $(+\infty)$	0	$\tan^{-1}\frac{(1-r)}{\delta}$ $\rightarrow0$ for N. H. and π for S. H. as $\gamma\rightarrow1$	Linear polarisation, Tilt$=0$	
	$\eta<0$, $(-\infty)$	0	Do.	Linear polarisation, Tilt$=0$	
$\xi\vert=\infty,\ \delta c=0$, $\delta\neq\infty$ (Pole)	$\eta<0$	1	$+\pi/2$	Circular polarisation	Anticlockwise for *o*-wave and Clockwise for *e*-wave
	$\eta<0$	1	$-\pi/2$	Do.	Clockwise for *o*-wave and Anticlockwise for *e*-wave
Ground	(N. H.)	$\sqrt{1+\delta c^2}-\vert\delta c\vert$		Ellipses, given by equation (2.17)	Sense of rotation shown in figs. (3) and (4)
Ground	(S. H.)	Do.		Do.	Do.

The axes of the polarisation ellipses are tilted with respect to x-axis through the angle ψ given by

$$\tan 2\psi^V = \frac{2R\cos\alpha}{1-R^2}, \quad \tan 2\psi^W = -\frac{2R\cos\alpha}{1-R^2}.$$

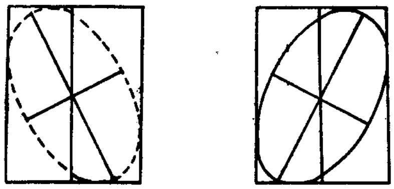

Fig. 6

It is easy to see that $\psi^V = (n+\frac{1}{2})\,\pi - \psi^W$. These ellipses can be shown to be inscribed within oblongs having the sides

$$\left(\pm\frac{C}{\sin\alpha}\right), \quad \left(\pm\frac{C}{R\sin\alpha}\right)$$

and touching the sides at the points

$$(\pm C_{V,W}/\sin\alpha),\ (\mp C_{V,W}\cot\alpha/R);\ (\pm C_{V,W}\cot\alpha),$$
$$\left(\mp C_{V,W}\frac{\sin\alpha}{R}\right). \qquad (2.17a)$$

The ratio of the axes is given by

$$\frac{(1+R^2)+\sqrt{(1+R^2)^2-4R^2\sin^2\alpha}}{(1+R^2)-\sqrt{(1+R^2)^2-4R^2\sin^2\alpha}}.$$

From these general expressions we can obtain the shape and sense of rotation of the polarisation ellipse at any point inside the ionosphere. We have described the nature of polarisation of the electric vector for any value of ξ,η in the last column of Table II.

§3. The Coupling Term

Let us now turn to the application of these formulae to actual cases. For these, we require the characteristics of ionospheric stations. These are given in Table III for a number of stations between the geomagnetic equator and N.M.P.

The second row gives the propagation angle θ of the stations given in the first row, and the other rows are self-explanatory. As we see from row 6, the value of ν_c, the magnetic damping factor varies from ∞ at the equator, to zero at the pole, passing through a value of 75×10^6/sec. at the ionospheric station nearest to the G.M.-equator, to $5{\cdot}5\times10^4$/sec. at the Clyde River station which is nearest to the N.M.P.

The values of all ionospheric quantities R, α, $\dot{\phi}$, q_o^2, q_e^2 are functions of ξ and η. Now $\xi=\nu/\nu_c$ and is independent of p. The collision frequency in any ionospheric layer is taken to vary as $\nu=\nu_0\exp\left\{-\frac{z-z_0}{l}\right\}$, where l is the scale height, ν_0 is the value of the collision frequency at the tip of the layer, z_0 is the height of the tip of the layer. The values of ν_0, z_0 vary with the hour of the day, the season and other factors. We have taken

$$\nu_0 \simeq 2.10^5/\text{sec. for the } E\text{-layer.}$$
$$\simeq 10^3/\text{sec. for the } F\text{-layer.}$$

as good average values for ν_0.

Table III

		Equator	Huancyao	Calcutta	Slough	Kiruna	Clyde River	North Pole
θ		90°	92°	112°	157°	167°	174°	180°
$\lvert\vec{H}\rvert$			$\cdot296\Gamma$	$\cdot434\Gamma$	$\cdot470\Gamma$	$\cdot514\Gamma$	$\cdot570\Gamma$	
$f_h = \left\lvert\frac{eH}{2\pi mc}\right\rvert$			$\cdot829{\cdot}10^6$	$1{\cdot}222{\cdot}10^6$	$1{\cdot}316{\cdot}10^6$	$1{\cdot}439.10^6$	$1{\cdot}596.10^6$	
$\Omega = \left\lvert\frac{\sin^2\theta}{2\cos\theta}\right\rvert$		∞	$14{\cdot}31$	$1{\cdot}148$	$\cdot083$	$\cdot026$	$\cdot005$	0
$\lvert\nu_c\rvert = 2\pi f_h\Omega$		∞	$74{\cdot}50.10^6$	$8{\cdot}79.10^6$	$\cdot686.10^6$	$\cdot235.10^6$	$\cdot055.10^6$	0
$\lvert\xi\rvert = \frac{\nu}{\lvert\nu_c\rvert}$;	$\nu = 2.10^5$	0	$\cdot269.10^{-2}$	$2{\cdot}27.10^{-2}$	$29{\cdot}2.10^{-2}$	$84{\cdot}8.10^{-2}$	364.10^{-2}	∞
	$\nu = 10^3$	0	$\cdot134.10^{-4}$	$1{\cdot}14.10^{-4}$	$14{\cdot}6.10^{-4}$	$42{\cdot}6.10^{-4}$	182.10^{-4}	∞
$\lvert\dot{\phi}\max\rvert$	$\nu = 2.10^5$, $l = 10$ km.	0	$\cdot200.10^{-3}$	$1{\cdot}71.10^{-3}$	$23{\cdot}9.10^{-3}$	227.10^{-3}	$22{\cdot}7.10^{-3}$	∞
	$\nu = 10^3$, $l = 50$ km.	0	$\cdot040.10^{-3}$	$\cdot341.10^{-3}$	$4{\cdot}37.10^{-3}$	$12{\cdot}8.10^{-3}$	$55{\cdot}6.10^{-3}$	∞

The corresponding values of ξ_0 are given in rows (7) and (8) for the E and F-layers.

We observe that for the F-layer, ξ_0 varies from $\simeq 10^{-4}$ at Huancayo to $1{\cdot}8\times 10^{-2}$ at Clyde River. We can therefore take $\xi << 1$ for F-layer propagation. At the G.M.E. $\xi \simeq 0$, and at the pole $\xi=\infty$. These points require separate treatment. For the treatment of wave propagation through F-layer, we have therefore to confine ourselves to sections I and IV of the $\xi\eta$-plane.

For the E-layer, ξ_0 continues to be small for low latitude stations, but at Slough it has attained the value ·29 and at times may approach unity. For the higher latitude stations, e.g. Clyde River $\xi_0 \simeq 3{\cdot}63$.

Path of the Ray in the $\xi\eta$-Plane

The vertical propagation of a ray in the $\xi\eta$-plane can be shown by a trajectory. We have

$$\xi=\xi_0 \exp\left\{-\frac{z-z_0}{l}\right\} \tag{3.1a}$$

and
$$\eta=\frac{1}{\delta_c}(1-r_0\gamma)$$

where $r_0=\dfrac{4\pi N_0 e^2}{mp^2}=p_c^2/p^2$ and

$$\gamma=\exp\frac{1}{2}\left[1-\frac{z-z_0}{l}-e^{-(z-z_0)/l}\right] \tag{3.1b}$$

is the Chapman Factor.

N_0 is the maximum concentration of ions in the layer.

Let us now consider the coupling coefficient $\dot\phi$. It can be easily shown that

$$\dot\phi=\tan^{-1}(\rho)=\frac{1}{2}\cdot\frac{\dot\eta-i\dot\xi}{(1-\xi^2+\eta^2)-2i\eta\xi} \tag{3.2}$$

We obtain after some work

$$R_e(\dot\phi)=\frac{1}{2}\cdot\frac{(1-\xi^2+\eta^2)\dot\eta+2\eta\xi\dot\xi}{(1-\xi^2+\eta^2)^2+4\eta^2\xi^2} \tag{3.3a}$$

$$I_m(\dot\phi)=-\frac{1}{2}\cdot\frac{(1-\xi^2+\eta^2)\dot\xi-2\eta\dot\eta\xi}{(1-\xi^2+\eta^2)^2+4\eta^2\xi^2} \tag{3.3b}$$

$$|\dot\phi|=\frac{1}{2}\left[\frac{\dot\eta^2+\dot\xi^2}{(1-\xi^2+\eta^2)^2+4\eta^2\xi^2}\right]^{\frac{1}{2}} \tag{3.3c}$$

Let us now find out values of $\dot\eta$ and $\dot\xi$. With the assumptions made in this section, i.e. $\nu=\nu_0\exp.[-(z-z_0)/l]$ it can be easily shown that

$$\dot\xi=\frac{d\xi}{du}=-\frac{c}{p}\frac{\xi}{l}$$

Similarly it can be shown, assuming that N is given by a Chapman-layer, that

$$\dot\eta=\frac{cr}{p\delta_c l}\beta=\frac{c}{pl}\beta\left(\frac{1}{\delta_c}-\eta\right)$$

where
$$\beta=\tfrac{1}{2}\left(1-e^{-(z-z_0)/l}\right)$$

We have therefore

$$R_e(\dot\phi)=\frac{c}{2pl}\cdot\frac{(1+\eta^2-\xi^2)\beta\left(\frac{1}{\delta_c}-\eta\right)-2\eta\xi^2}{(1+\eta^2-\xi^2)^2+4\eta^2\xi^2} \tag{3.4a}$$

$$I_m(\dot\phi)=\frac{c}{2pl}\cdot\frac{(1-\xi^2+\eta^2)\xi+2\eta\xi\beta\left(\frac{1}{\delta_c}-\eta\right)}{(1+\eta^2-\xi^2)^2+4\eta^2\xi^2} \tag{3.4b}$$

$$|\dot\phi|=\frac{c}{2pl}\cdot\left[\frac{\beta^2\left(\frac{1}{\delta_c}-\eta\right)^2+\xi^2}{(1+\eta^2-\xi^2)^2+4\eta^2\xi^2}\right]^{\frac{1}{2}} \tag{3.4c}$$

From the above expression for $|\dot\phi|$ it is seen that this attains its maximum value at $\eta=0$. We have then

$$|\dot\phi_{\max}|=\frac{c}{2pl}\cdot\left[\frac{\beta^2/\delta_c^2+\xi^2}{(1-\xi^2)^2}\right]^{\frac{1}{2}} \tag{3.5}$$

Therefore $|\dot\phi_{\max}|$ is very large for $\xi\simeq 1$ and tends to infinity as $\xi\to 1$.

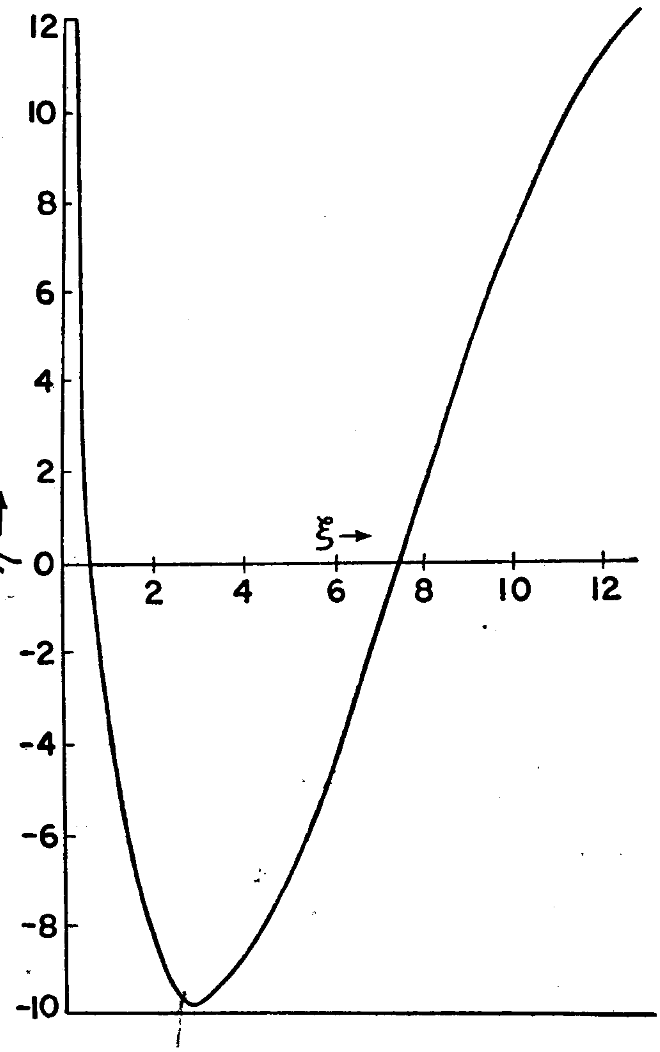

Fig. 7

When a wave of any frequency travels through a layer we observe that to every point of the layer there correspond definite values of ξ and η; we can therefore plot on the $\xi\eta$-plane a series of points corresponding to different values of z. The curves joining these points may be called the trajectory of the wave in the $\xi\eta$-plane. Such a trajectory drawn for Slough is shown in fig. 7. The points ξ and η have been calculated on the assumption that the variation of collision frequency is given by (3.1*a*) and the electron concentration is given by (3.1*b*). As we are interested in the maximum value of $\dot{\phi}$, we require the value of ξ at $\eta=0$ on the trajectory of the wave in the $\xi\eta$-plane. This value of ξ depends on the frequency of the wave and the location of the station. Fig 8 gives a number of (ξ, θ) curves (at $\eta=0$) for different values of p/p_c. The layer has been assumed to be the E-layer with $\nu_0=2.10^5$/sec. The corresponding values of ξ for an F-layer with $\nu_0=10^3$/sec. will simply be 1/200th of the values given by the curves.

Certain conclusions can be immediately reached from the nature of these curves.

The frequency at which the point $(\xi=1, \eta=0)$ is crossed depends on the magnetic characteristics of the station and the relation is given as

$$p=p_c(\nu_c/\nu_0)^{\frac{1}{2}}e^{-\frac{1}{2}\nu_c/\nu_0}$$

For stations where the magnetic damping is large compared to the collisional damping (equatorial region) this point is reached when $p<<p_c$, while for high latitude stations $\nu_c<<\nu_0$ and this critical value of p approaches p_c. For F-layer propagation, since $\nu_c>>1$ practically over the entire globe (excepting the small polar belt), it follows that the critical point $(\xi=1, \eta=0)$ will be reached when $p<<p_c$, i.e. when the wave will fail to reach F-layer owing to the presence of the lower E-layer.

Only when $\nu_c/\nu_0\simeq1$ cases of practical importance will arise. Ionospheric stations like Kiruna and Slough are

TABLE OF NOTATIONS

Quantity	Saha *et al.*	Appleton *et al.*	Hartree	Eckersley	Rydbeck
Direction of Propagation	z	z	..	z	z
Horizontal in Magnetic Meridian	x	x	..	x	y
Horizontal ⊥ Magnetic Meridian	y	y	..	y	x
Propagation Angle	θ	..	..	..	θ_g
Field Vectors	E,H,P,D	E,H,P,D	L,H,D	$E,P,$	E,D,P
Earth's Magnetic Field	H	H	H	H	H
Refractive Index	μ	μ	μ	μ	μ
Complex Refractive Index	$q=\mu-ick/p$	cq	K	Z	$\sqrt{\epsilon}$
Absorption Coefficient	K	K	..	..	..
Pulsatance	p	p	kc	$2\pi\nu$	ω
Gyropulsatance $\frac{eH}{mc}$	p_h	ph	k_hc	$2\pi\nu_H$	ω_H
Frequency	f	f	$kc/2\pi$	ν	$\omega/2\pi$
Gyrofrequency	f_h	f_h	$k_hc/2\pi$	ν_H	$\omega_H/2\pi$
Collision Frequency	ν	ν	$2k_fc$	ν_c	ν
Relative Gyrofrequency $f_h/f=p_h/p$	ω	$-\gamma/, \alpha, y$	τ	τ	γ/x_0^2
Relative Collision Frequency ν/p	δ	..	..	$\alpha_0/2\pi$	δ
Phase Velocity	v	v	..	..	..
Group Velocity	w	w	..	..	..
Displacement of the Ion	$\vec{S}(\xi, \eta, \zeta)$	x,y,z	$\vec{P}$	..	..
Dipole Moment	$Ne\vec{S}$	$Ne(x,y,z)$	$Ne\vec{P}$	..	..
Scattering Tensor	$\frac{\gamma\Delta}{\beta(\beta^2-\omega^2)}$	..	σ	$-\alpha$	..
Ratio of Axes	ρ	R	..	..	u
Critical Pulsatance $\frac{4\pi Ne^2}{m}$	p_o^2	p_o^2	..	$4\pi^2\nu_0^2$	ω_o^2
$4\pi Ne^2/mp^2=p_0^2/p^2$	r	$-\frac{1}{\alpha}, x$	σ_0	ζ	$\frac{1}{x_0^2}$
$\frac{eH}{mcp}\Big/[1-i\nu/p]$	$-\omega/\beta$	..	$K\alpha H$	ν_H/ν'	..
$\frac{4\pi Ne^2}{mp^2}\Big/(1-i\nu/p)\,[1-p_h^2/p^2\,(1-i\nu/p)^2]$	$-4\pi A/\beta^2$	$\frac{\alpha}{\alpha^2-y^2}$	ξ	ζ	..
$\frac{4\pi Ne^2}{mp^2}\Big/[1-i\nu/p]$	r/β	$\frac{1}{\alpha+\alpha}$	σ_0	..	..
$mp\nu/4\pi Ne^2$	δ/r	β	..	..	..
$\frac{mp}{4\pi Ne^2}\left[\frac{eH}{mc}\right]$	$\frac{\omega}{r}$	γ	P/σ_0	..	γ

therefore suitable for observations of any peculiarities arising out of this coupling term. In such cases $|\dot{\phi}|_{max}$ will be quite large and the approximate differential equations

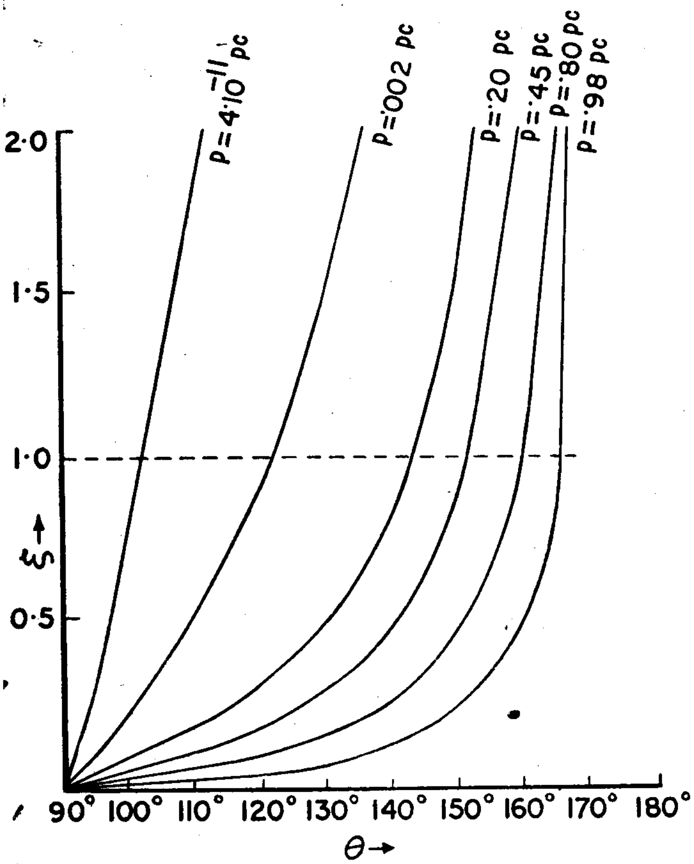

Fig. 8

will no longer be valid. Though this does not necessarily imply that the large $|\dot{\phi}|$ is solely responsible for the triple splitting, as has been suggested by Rydbeck, it cannot be denied that the nature of propagation may be profoundly modified. As no one has yet been able to give an exact treatment of the differential equations, it is not safe to make any definite statement about the nature of this modification.

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86. OCCURRENCE OF STRIPPED NUCLEI OF NEON IN PRIMARY COSMIC RAYS

(*Nature*, **167**, 476, 1951)

Bradt and Peters, in their analysis of the primary cosmic radiation as observed in the out-of-the-atmosphere observations with the plate technique, have given the completely stripped nucleus of neon as one of the main components of the heavier cosmic particles. In fact, the relative abundance is given as almost the same as that of oxygen-16 (*vide* Fig. 13, p. 66, of their paper).

It appears that if the identification of the stripped nucleus of neon as one of the main constituents of primary cosmic particles be correct, and is confirmed by subsequent observations, it constitutes a very strong argument against the hypothesis that the sun is the source of cosmic particles received on the earth[2,3]. For to have stripped nuclei of neon from the sun, it must be first demonstrated that neon exists on the sun and is at least once ionized on the photosphere or the chromosphere. The evidence on these points, as will be shown presently, is absolutely negative, in spite of the fact that strong lines of Ne and Ne^{+} occur within the solar range of wave-lengths (3,000-10,000 A.)