Lectures on
Perturbative Renormalization Group
(Draft)

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CHAPTER 1

Introduction

Speaking in 2004, there are many excellent introductions to Renormalization Group (RG). First of all, there is the popular article by K.G. Wilson [1] and four technical review articles [2, 3, 4, 5] on the so-called Wilson RG. There is the article by Weinberg [6] on perturbative RG. Almost all modern quantum field theory books have at least one chapter on perturbative RG, for example, Cheng and Li [7]. Text books on Quantum Chromodynamics also unavoidably discuss RG in detail, for example, Muta [8]. Older text books on quantum field theory discuss perturbative RG of Gell-Mann-Low [9] variety. Modern books do discuss the difference between perturbative RG and Wilson RG. In this regard, one would especially mention Peskin and Schroder [10], Weinberg [11] and Zee [12]. Our treatment of RG in the context of broken scale invariance follows Wilson which also appears in the textbook of Pokorski [13]. (It goes without saying that any omission of relevant references here is just a display of my ignorance.)

There are also textbooks specialized to RG. For RG in the context of the so called ε - *expansion* the text book by Amit [14] may be consulted. For conceptual clarity on topics like anomalous dimension, application of RG to classical phenomena, etc. the book by Goldenfeld [15] is highly recommended.

Then why another set of lecture notes on perturbative RG? Definitely I cannot claim any originality for any of the topics discussed here. The only novelty is in the particular selection of materials (all taken from the research papers, review articles and books) and the order of presentation. The hope is that after going through the elementary materials *collected* here, the reader gets enough motivation to plunge into the original articles and cited books. Then the reader may eventually resolve all the seemingly contradictory statements on RG by the Masters in the field. (A sample of this is accessible on the internet at

`http://www.saha.ac.in/theory/a.harindranath/RG-fun.html`

for browsing pleasure.) Hopefully this justifies yet another set of lecture notes on perturbative Renormalization Group. These notes are based on four lectures delivered in the Theory Group, SINP, Kolkata in the period December 2003 - January 2004. I thank the participants for asking probing questions. As a result, the written version is much better than the spoken version. I thank Dipankar Chakrabarti, Ramesh Babu Thayyullathil and Santanu Mondal for careful reading of the previous versions of the manuscript and providing typographical corrections.

CHAPTER 2

Scale Invariance: Canonical Considerations

2.1 Scale symmetry in classical mechanics

A discussion of scale symmetry in classical mechanics can be found in Ref. [16, 17]. In this section, we follow these references.

For simplicity, without loss of generality, we work in 1+1 dimensions. Consider the action describing a free particle of unit mass:

$$S = \int dt \frac{1}{2}\dot{x}^2 \ . \tag{2.1}$$

Consider the scale transformation which is a dilation (dilatation) of time

$$t \to t' = s^{-1}t \ . \tag{2.2}$$

We find that the action remains invariant under transformation provided

$$x \to s^{-\frac{1}{2}}x \ . \tag{2.3}$$

Note that with the potential $V(x) = -\frac{1}{2}\frac{1}{x^2}$, the action

$$S = \int dt \left[\frac{1}{2}\dot{x}^2 + \frac{1}{2}\frac{1}{x^2}\right] \tag{2.4}$$

and the equation of motion

$$\frac{d^2x}{dt^2} - \frac{1}{x^3} = 0 \tag{2.5}$$

remains invariant under

$$t \to s^{-1}t, \quad x \to s^{-\frac{1}{2}}x \ . \tag{2.6}$$

Note that from $p = \dot{x}$, scaling law for the momentum p is $p \to s^{\frac{1}{2}}p$. Then the Hamiltonian

$$H = \frac{1}{2}p^2 - \frac{1}{2}\frac{1}{x^2} \quad \to sH \ . \tag{2.7}$$

The Hamiltonian scales, (it is important to note that the Hamiltonian is *not* invariant). From the scaling laws, one may assign scale dimensions $-\frac{1}{2}$, $\frac{1}{2}$ and 1 to x, p and H respectively.

Define

$$D = -\frac{1}{2}xp + Ht. \tag{2.8}$$

(See Goldstein, Classical Mechanics, Second Edition, page 430, problem 2.)

Using $\frac{dx}{dt} = p$ and $\frac{dp}{dt} = -\frac{1}{x^3}$, we get $\frac{dD}{dt} = 0$. Thus D is a constant of motion. Note that we are dealing with a dynamical symmetry since H is not invariant under the transformation. Also note that at $t = 0$, $D = -\frac{1}{2}xp$.

We find

$$\begin{aligned}
\delta_D\, x &= [x, D]_{PB} &&= -t\frac{dx}{dt} - \frac{1}{2}x \\
\delta_D\, p &= [p, D]_{PB} &&= t\frac{dp}{dt} + \frac{1}{2}p \\
\delta_D\, H &= [H, D]_{PB} &&= H.
\end{aligned} \tag{2.9}$$

Note that the scale dimension $-\frac{1}{2}$, $\frac{1}{2}$ and 1 appear as coeffcients of x, p and H in these relations.

Note also that the Hamiltonians containing familiar potentials like the Coulomb potential or the Harmonic oscillator potential won't exhibit scaling behaviour since the kinetic term and the potential term will have different scaling behaviour. One can also easily check that the equations of motion in these cases won't be invariant under the transformations, $x \to s^{-1/2}x$, $t \to s^{-1}t$.

From the discussion, it should be clear that the scale invariance associated with $\frac{1}{x^2}$ potential will also hold in 2+1 and 3+1 dimensions. Further in 2+1 dimensions, the delta function potential $\delta^2(\mathbf{r})$ also will be scale invariant. The later potential is a laboratory to illustrate asymptotic freedom and dimensional transmutation in quantum mechanics. For more on this, see for example Jackiw [18] and Perry [19].

2.2 Short distance behaviour and anomalous dimension in quantum mechanics

The concept of *anomalous dimension* could be understood from elementary quantum mechanics following de Alfaro, Fubini, Furlan and Rossetti [20]. Consider Schroedinger equation

$$\left[-\frac{\hbar^2}{2\mu}\Delta + V(r)\right]\phi(\vec{r}) = E\phi(\vec{r}) \tag{2.10}$$

with

$$\begin{aligned}
\Delta &= \frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2} + \frac{1}{tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \\
&= \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{\vec{L}^2}{r^2}.
\end{aligned} \tag{2.11}$$

Writing

$$\phi(\vec{r}) = R(r)\, Y_{lm}(\theta, \phi) \tag{2.12}$$

with

$$\vec{L}^2\phi(\vec{r}) = l(l+1)\phi(\vec{r}), \tag{2.13}$$

$$\left[\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}V(r) + \frac{2\mu}{\hbar^2}E\right]u(r) = 0 \tag{2.14}$$

where we have put $R(r) = \frac{u(r)}{r}$. Thus

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{G(r)}{r^\eta}\right]u(r) = 0 \tag{2.15}$$

where we have put $k^2 = \frac{2\mu}{\hbar^2}E$ and $\frac{G(r)}{r^\eta} = \frac{2\mu}{\hbar^2}V(r)$.

Thus we have the free kinetic part $\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}$, 'mass' term k^2 and interaction term $\frac{G(r)}{r^\eta}$.

Let $Limit_{r\to 0}G(r) = G^2$.

Case I. $\eta < 2$. "super renormalizable". For example, for $\eta = 1$, G^2 has mass dimension.

We are interested in the behaviour near origin $r \to 0$ (short distance).

Kinetic term dominates at small r. For $r \to 0$, we can put $u(r) = r^{s+1}$. From $\frac{d^2u}{dr^2} - \frac{l(l+1)}{r^2}u = 0$ we get $s = l$ or $s = -(l+1)$. The requirement that acceptable solution should go to zero at origin for all l, means that we accept only the solution $s = l$.

Thus limit $r \to 0$, $u(r) = r^{l+1}$.

Define the scale dimension to be the eigenvalue of the operator $D = r\frac{d}{dr}$. Then $u(r)$ has scale dimension $l+1$. Note that, in retrospect, we have rejected the solution with negative scale dimension.

Even though the complete solution is not an eigenfunction of D and scale breaking enters through dependence on k^2 and G^2, short distance behaviour of the wavefunction is that of free field theory. Scale dimension is canonical.

Case II. $\eta = 2$. "just renormalizable".

Write

$$\bar{l}(\bar{l}+1) = l(l+1) + G^2. \tag{2.16}$$

Then, limit $r \to 0$, $u(r) = r^{\bar{l}+1}$ where $\bar{l} + \frac{1}{2} = \sqrt{(l+\frac{1}{2})^2 + G^2}$. Now the scale dimension is "dynamical" (it depends on the interaction) but we can still talk of solution with a given scale dimension. We see that the interaction " renormalizes" the canonical scale dimension to anomalous dimension.

Case III. $\eta > 2$. Short distance behaviour is dominated by the interaction, the concept of scale dimension is no longer useful. This case is left as a homework problem for the reader.

Note that case I is close to super renormalizable theories like ϕ^3 interaction in 3+1 dimensions in quantum field theory. Case II is like ϕ^4 interaction in 3+1 dimensions.

2.3 Scale Symmetry in classical field theory

Here we follow standard text book treatment, for example, Ref. [21].

Consider the scale transformation $x \to x' = sx$. Under this transformation, let the classical filed $\phi(x) \to \phi'(x') = s^{-d_\phi}\ \phi(x)$. s is the scale (dilation) factor and d_ϕ is the scale dimension of the field $\phi(x)$.

Consider the action

$$\mathscr{S} = \int d^d x\, \mathscr{L}(\phi(x), \partial^\mu \phi(x)). \tag{2.17}$$

The transformed action

$$\mathscr{S}' = \int d^d x'\ \mathscr{L}(\phi'(x'), \partial'_\mu \phi'(x')). \tag{2.18}$$

We have $\partial'_\mu = s^{-1}\partial_\mu$. The Jacobian of transformation

$$J = \left| \frac{\partial x'}{\partial x} \right| = s^d \tag{2.19}$$

where d is the dimension of space-time. Then

$$\mathscr{S}' = s^d \int d^d x\, \mathscr{L}(s^{-d_\phi}\phi(x), s^{-1-d_\phi}\partial_\mu \phi(x)). \tag{2.20}$$

Explicitly, for,

$$\mathscr{S} = \int d^d x\, \partial_\mu \phi\ \partial^\mu \phi \tag{2.21}$$

$\mathscr{S}' = \mathscr{S}$ if $d_\phi = \frac{d}{2} - 1$. We can add a term ϕ^n to the Lagrangian density, provided $n\ d_\phi = d$ or $n = \frac{2d}{d-2}$. Thus for $d = 6$, $n = 3$, for $d = 4$, $n = 4$, for $d = 3$, $n = 6$ and lastly for $d = 2$, $n \to \infty$, any integer n is allowed!

2.4 Scale symmetry in quantum field theory

Here we follow the treatment in Pokorski [13].

Consider scale transformation defined as $x \to x' = sx = e^\varepsilon\ x$. Let us consider the corresponding transformation on classical fields. For simplicity we consider a scalar field which carry no Lorentz index:

$$\phi(x) \to \phi'(x') = T(\varepsilon)\ \phi(x) = (\exp\varepsilon)^{-d_\phi}\phi(x). \tag{2.22}$$

Let us denote the state vector in the Hilbert space by $\mid v\rangle$. We have $\langle x \mid = \langle \Omega \mid \Phi(x)$ where $\mid \Omega\rangle$ is the vacuum state and $\Phi(x)$ is the field operator. The classical field, or the wave function

$$\phi(x) = \langle x \mid v\rangle = \langle \Omega \mid \Phi(x) \mid v\rangle. \tag{2.23}$$

Now, $\langle x' \mid = \langle \Omega \mid \Phi(x')$ and let $\mid v'\rangle = U^\dagger(s) \mid v\rangle$. We get

$$\phi'(x') = \langle x' \mid v'\rangle = \langle \Omega \mid \Phi(x') U^\dagger(s) \mid v\rangle = T\phi(x) \tag{2.24}$$

using the invariance of the vacuum $\langle\Omega\,|\,U^\dagger(s) = \langle\Omega\,|$. Thus $T\Phi(x) = U(s)\Phi(x')U^\dagger(s)$ or $U^\dagger(s)\Phi(x)U(s) = T^{-1}\,\Phi(x')$. Thus we conclude that, as

$$x \to x' = sx, \tag{2.25}$$

the classical field transforms as

$$\phi(x) \to \phi'(x') = (\exp\,\varepsilon)^{-d_\phi}\;\phi(x) \tag{2.26}$$

and consequently the quantum field operator transforms as

$$\Phi(x) \to \Phi'(x) = U^\dagger(s)\Phi(x)U(s) = s^{-d_\phi}\;\Phi(sx) \tag{2.27}$$

with $s = \exp\,\varepsilon$.

2.4.1 Canonical considerations

Here we follow Wilson [22, 23].

We have seen that in quantum field theory, the scale transformation on a generic field $\Phi(x)$ is achieved by means of a unitary transformation

$$\Phi(x) \to \Phi'(x) = U^\dagger(s)\Phi(x)U(s) = s^{d_\phi}\Phi(sx). \tag{2.28}$$

Canonically the scale dimension is determined by insisting that the canonical commutation relation is invariant under scale transformation.

For example, consider the fermion field $\psi(x)$ which obeys canonical (anti-)commutation relation

$$\{\psi(x), \psi^\dagger(y)\}_{x^0=y^0} = \delta^3(x-y). \tag{2.29}$$

Under the scale transformation $x \to sx$, $\psi(x) \to U^\dagger(s)\psi(x)U(s) = \psi'(x)$. Let us first motivate why the factor s^d is introduced. Suppose we take $\psi'(x) = \psi(sx)$. It is easily verified that $\psi'(x) = \psi(sx)$ does not satisfy canonical commutation relation. Then we define $\psi'(x) = s^{d_\psi}\psi(sx)$ and choose d_ψ so that $\psi'(x)$ satisfy canonical commutation relation. This gives $d_\psi = 3/2$ which incidentally is the same as the engineering dimension of ψ in 3+1 dimensions.

Next we consider scalar field. *For simplicity we denote the classical and quantum field by the same symbol ϕ.*

Consider the free scalar field Lagrangian density

$$\mathscr{L}_0 = \frac{1}{2}\,\partial^\mu\phi\;\partial_\mu\phi. \tag{2.30}$$

The action

$$\mathscr{S}_0 = \int d^4x\,\mathscr{L}_0. \tag{2.31}$$

The equation of motion is

$$\partial^\mu \partial_\mu \phi = 0. \tag{2.32}$$

Equal time commutation relation is

$$[\phi(x,t), \frac{d\phi(y,t)}{dt}] = i\delta^3(x-y). \tag{2.33}$$

The Hamiltonian is

$$H = \frac{1}{2}\int d^4x \left[(\frac{\partial\phi}{\partial t})^2 + (\nabla\phi)^2 \right]. \tag{2.34}$$

If $\phi(x)$ is a solution of the equation of motion, Eq. (2.32), it is readily verified that $\phi(sx)$ is also a solution. The field $\phi(sx)$ however does not satisfy the canonical commutation relation given in Eq. (2.33). On the other hand $\phi'(x) = s^{d_\phi}\ \phi(sx)$ with $d_\phi = 1$ satisfy the canonical commutation relation and also is a solution to the equation of motion. From the Heisenberg equation of motion

$$i\frac{\partial\phi(x,t)}{\partial t} = [\phi(x,t), H] \tag{2.35}$$

we find that if $U^\dagger(s)HU(s) = sH$, $\phi(sx)$ obey the correct equation of motion, namely, $i\frac{\partial\phi(sx,st)}{\partial st} = [\phi(sx,st), H]$.

2.4.2 Generator of scale transformation and the divergence of scale current

Let us write $U(s) = e^{-i \ln s\, D}$. For infinitesimal transformation put $s = 1+\varepsilon$ so that $\ln s = \varepsilon$. From $U^\dagger(s)\phi(x)U(s) = s^d\ \phi(sx)$, for infinitesimal transformation we get

$$i[D, \phi(x)] = [d + x_\mu \partial^\mu]\phi(x) = \delta\phi(x). \tag{2.36}$$

With $\phi(x) \to \phi(x) + \varepsilon\ \delta\phi(x)$ let us compute the variation in the Lagrangian density

$$\delta\mathscr{L} = \mathscr{L}(\phi + \delta\phi) - \mathscr{L}(\phi) \tag{2.37}$$

For $\mathscr{L}_0$ given in Eq. (2.30), we find $\delta\mathscr{L} = [4 + x_\rho\partial^\rho]\mathscr{L}_0$. This shows that $\mathscr{L}_0$ has scale dimension 4. The Lagrangian density does not remain invariant but the action remains invariant since

$$\delta\mathscr{S} = \int d^4x\ \delta\mathscr{L}_0 = 0. \tag{2.38}$$

Next consider $\mathscr{L}_s = -\frac{\lambda}{4!}\phi^4$. We find $\delta\mathscr{L}_s = [4 + x_\rho\partial^\rho]\mathscr{L}_s$. This shows that $\mathscr{L}_s$ also has scale dimension 4. Thus $\int d^4x\ \delta\mathscr{L}_s = 0$.

Lastly consider $\mathscr{L}_B = -\frac{1}{2}m^2\phi^2$. We get $\delta\mathscr{L}_B = [2 + x_\rho\partial^\rho]\mathscr{L}_s$. Thus $\mathscr{L}_B$ has scale dimension 2.

We have

$$\int d^4x\ \delta\mathscr{L}_B = \mu^2 \int d^4x\ \phi^2. \tag{2.39}$$

Thus mass term in the Lagrangian density $\mathscr{L} = \frac{1}{2}\,\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$ violates scale invariance.

Next we identify the scale current (also called dilatation current) and calculate the divergence of the scale current. In a scale invariant theory we found $\delta\ \mathscr{L} = [4 + x_\mu\partial^\mu)]\mathscr{L} = \partial^\mu(x_\mu\mathscr{L})$ without using the Euler-Lagrange equation of motion. On the other hand, using explicitly the Euler-Lagrange equation of motion,

$$\delta\mathscr{L} = \mathscr{L}\Big(\phi + \delta\phi, \partial_\mu(\phi + \delta\phi)\Big) - \mathscr{L}\Big(\phi, \partial_\mu\phi\Big) = \partial_\mu(\pi^\mu\delta\phi) \tag{2.40}$$

with $\pi^\mu = \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}$. Thus we arrive at

$$\partial^\mu D_\mu = \partial^\mu(\pi_\mu\delta\phi - x_\mu\mathscr{L}) = 0. \tag{2.41}$$

Thus we identify the dilatation (or dilation) current

$$D_\mu = \pi_\mu\ \delta\phi - x_\mu\mathscr{L}. \tag{2.42}$$

Starting from $D_\mu = \pi_\mu\ \delta\phi - x_\mu\ \mathscr{L}$ we compute the divergence of the scale current

$$\begin{aligned}\partial^\mu D_\mu &= -4\mathscr{L} - x_\mu\partial^\mu\mathscr{L} + \pi_\mu\partial^\mu\delta\phi + \partial^\mu\pi_\mu\delta\phi \\ &= -4\mathscr{L} + \pi^\mu\partial_\mu\phi + \partial^\mu(\pi_\mu\phi)\end{aligned} \tag{2.43}$$

using $\delta\phi = \phi + x_\rho\partial^\rho\phi$ and $\partial^\mu\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi}\,\partial^\mu\phi + \frac{\partial\mathscr{L}}{\partial\partial_\rho\phi}\,\partial^\mu\partial_\rho\phi$.

From the canonical energy momentum tensor $\theta^\mu_{\ \nu} = \pi^\mu\ \partial_\nu\phi - g^\mu_{\ \nu}\mathscr{L}$, we get the trace of the energy momentum tensor $\theta^\mu_{\ \mu} = -4\mathscr{L} + \pi^\mu\partial_\mu\phi$. Thus, for the scalar field, we arrive at

$$\partial^\mu D_\mu = \theta^\mu_{\ \mu} + \partial^\mu(\pi_\mu\phi) = \mu^2\phi^2. \tag{2.44}$$

2.4.3 Why scale symmetry is a bad symmetry

For translation

$$U(a) = e^{-ia^\mu P_\mu} \tag{2.45}$$

and for scale transformation

$$U(\varepsilon) = e^{-i\varepsilon D}. \tag{2.46}$$

For infinitesimal transformation, consider $U^\dagger(a)U^\dagger(\varepsilon)\phi(x)U(\varepsilon)U(a) - U^\dagger(\varepsilon)U^\dagger(a)\phi(x)U(a)U(\varepsilon)$. On the one hand we get, $e^{d\varepsilon}\Big(\phi(e^\varepsilon x + a) - \phi(e^\varepsilon(x + a)\Big) = -\varepsilon a^\mu\partial_\mu\phi = -\varepsilon a^\mu\ i\ [P_\mu, \phi]$. On the operator side we get, $\varepsilon\ a^\mu\ \Big[[D, P_\mu], \phi\Big]$. Thus we get

$$i[D, P_\mu] = P_\mu. \tag{2.47}$$

From Eq. (2.47) it immediately follows that

$$[D, P^\mu P_\mu] = -2\, i\, P^2. \tag{2.48}$$

Exponentiating

$$e^{i\varepsilon D} P^2 e^{-i\varepsilon D} = e^{2\varepsilon}\, P^2. \tag{2.49}$$

Let $\mid p\rangle$ be an eigenstate of P^2 with eigenvalue p^2. Then $e^{i\varepsilon\, D} P^2\, e^{-i\varepsilon\, D}\ \mid p\rangle = e^{2\varepsilon}\, p^2 \mid p\rangle$. Thus we conclude that if $\mid p\rangle$ is an eigenstate of P^2 with eigenvalue p^2, $e^{-i\varepsilon\, D}\ \mid p\rangle$ is an eigenstate of P^2 with eigenvalue $e^{2\varepsilon}\ p^2$. This means that either all the states are massless or the spectrum is continuous. This is a most unwelcome result since this rules out mass gaps and hence bound states.

Thus scale invariance, unlike Lorentz symmetry or Gauge symmetry is an *unwanted* symmetry. There are three ways this symmetry may be broken:

- ☞ *Explicit breaking*. Example: With a term $-\frac{1}{2}\mu^2\phi^2$ in the Lagrangian density, we got $\partial^\mu D_\mu = \mu^2\phi^2$.
- ☞ *Spontaneous symmetry breaking*. Vacuum breaks the symmetry, i.e., $e^{-i\varepsilon\, D} \mid \Omega\rangle \neq\mid \Omega\rangle$. We have Nambu-Goldstone realization of symmetry with the emergence of the Goldstone boson (dilaton).
- ☞ *Anomalous breaking*. Classical symmtery broken by quantum effects - visible in perturbation theory.

We will study scale symmetry breaking of the first and third kind.

CHAPTER 3

Naive Ward Identity of Broken Scale Invariance

In this chapter, following Wilson [22, 23] (see also Callan [24] and Symanzik [25]) we derive the *naive* Ward identity of broken scale invariance. See also chapter 7 of Pokorski [13]. The whole content of this chapter is a *trivial exercise in Fourier transforms* according to Coleman [26].

3.1 Naive Ward identity in coordinate space

When we have exact scale invariance, $\partial^\mu D_\mu = 0$ where D_μ is the scale current. When scale invariance is broken explicitly by the presence of parameters having dimensions of mass in the Lagrangian density, the scale current acquires a non vanishing divergence, $\partial^\mu D_\mu(x) = S(x)$. (For scalar field theory, we saw, $S(x) = m^2\phi^2$.

Let $\mid \Omega\rangle$ denote the vacuum state. We wish to consider the matrix element

$$\tilde{M}^{(n)}(x_1,x_2,\ldots,x_n) = \int d^4y\ \langle\Omega \mid T\Big[\phi(x_1)\phi(x_2)\ldots\phi(x_n)S(y)\Big] \mid \Omega\rangle > \tag{3.1}$$

where T is the time ordering symbol. Thus $\tilde{M}^{(n)}$ is the n-point Greens function with the zero momentum insertion $\int d^4y\ S(y)$. Now we are going to perform a set of canonical manipulations that are usually done to derive Ward Identities that arise as a consequence of some conserved current (or softly broken symmetry). To show the manipulations explicitly, we specialize to the case $n = 2$ for simplicity.

$$\begin{aligned}\tilde{M}^{(2)}(x_1,x_2) &= \int d^4y\ \langle\Omega \mid T\Big[\phi(x_1)\phi(x_2)S(y)\Big] \mid \Omega\rangle \\ &= \int d^4y\ \langle\Omega \mid \Big\{\theta(x_1^0-x_2^0)\theta(x_2^0-y^0)\phi(x_1)\phi(x_2)S(y) \\ &\qquad +5\ \text{ more terms}\Big\} \mid \Omega\rangle.\end{aligned} \tag{3.2}$$

Now replace $S(y)$ by $\partial_\mu^y D^\mu(y)$. We have put a superscript on the derivative to remind us that the derivative is with respect to y. Do a partial integration and ignore the surface terms. Then

$$\begin{aligned}\tilde{M}^{(2)}(x_1,x_2) &= -\int d^4y\ \langle\Omega \mid \Big\{\Big[\partial_\mu^y\theta(x_2^0-y^0)\Big]\theta(x_1^0-x_2^0)\phi(x_1)\ \phi(x_2)\ D^\mu(y) \\ &\qquad +5\ \text{ more terms}\Big] \mid \Omega\rangle.\end{aligned} \tag{3.3}$$

Noting that the derivative will involve only the time derivative, we replace

$$\int d^3y\, D^0(y^0,y) = D(y^0) \tag{3.4}$$

where $D(y^0)$ is the charge or the generator of scale transformation. Then

$$\begin{aligned}\tilde{M}^{(2)}(x_1,x_2) &= -\int dy^0\, \langle\Omega \mid \left\{ \left[\frac{\partial}{\partial y^0}\theta(x_2^0-y^0)\right]\phi(x_1)\;\phi(x_2)\;D(y^0)+5\;\;\text{more terms}\right\} \mid \Omega\rangle \\ &= \langle\Omega \mid \left\{\phi(x_1)\;\phi(x_2)\;D(x_2^0)+7\;\;\text{more terms}\right\} \mid \Omega\rangle > \end{aligned}\tag{3.5}$$

i.e.,

$$\begin{aligned}\tilde{M}^{(2)}(x_1,x_2) &= \theta(x_1^0-x_2^0)\langle\Omega \mid \phi(x_1)\Big[\phi(x_2),D(x_2^0)\Big] \mid \Omega\rangle \\ &\quad+\theta(x_1^0-x_2^0)\langle\Omega \mid \Big[\phi(x_1),D(x_1^0)\Big]\phi(x_2)\Big] \mid \Omega\rangle \\ &\quad+\theta(x_2^0-x_1^0)\langle\Omega \mid \phi(x_2)\Big[\phi(x_1),D(x_1^0)\Big] \mid \Omega\rangle \\ &\quad+\theta(x_2^0-x_1^0)\langle\Omega \mid \Big[\phi(x_2),D(x_2^0)\Big]\phi(x_1) \mid \Omega\rangle.\end{aligned}\tag{3.6}$$

Using

$$\Big[\phi(x_1),D(x_1^0)\Big] = i[d+x_{1\mu}\partial_1^{\mu}]\phi(x_1) \tag{3.7}$$

$$\begin{aligned}\tilde{M}^{(2)}(x_1,x_2) &= 2id\langle\Omega \mid T[\phi(x_1)\phi(x_2)] \mid \Omega\rangle + \\ &\quad+i\theta(x_1^0-x_2^0)\langle\Omega \mid \phi(x_1)x_{2\mu}\partial_2^{\mu}\phi(x_2) \mid \Omega\rangle \\ &\quad+i\theta(x_1^0-x_2^0)\langle\Omega \mid x_{1\mu}\partial_1^{\mu}\phi(x_1)\phi(x_2) \mid \Omega\rangle \\ &\quad+i\theta(x_2^0-x_1^0)\langle\Omega \mid \phi(x_2)x_{1\mu}\partial_1^{\mu}\phi(x_1) \mid \Omega\rangle \\ &\quad+i\theta(x_2^0-x_1^0)\langle\Omega \mid x_{2\mu}\partial_2^{\mu}\phi(x_2)\phi(x_1) \mid \Omega\rangle.\end{aligned}\tag{3.8}$$

Using

$$\begin{aligned}x_{2\mu}\partial_2^{\mu}\Big[\theta(x_1^0-x_2^0)\langle\Omega \mid \phi(x_1)\phi(x_2) \mid \Omega\rangle\Big] &= \\ \theta(x_1^0-x_2^0)\langle\Omega \mid \phi(x_1)x_{2\mu}\partial_2^{\mu}\phi(x_2) \mid \Omega\rangle & \\ +x_{2\mu}\Big[\partial_2^{\mu}\theta(x_1^0-x_2^0)\Big]\langle\Omega \mid \phi(x_1)\phi(x_2) \mid \Omega\rangle &\end{aligned}\tag{3.9}$$

we get

$$\tilde{M}^{(2)}(x_1,x_2) = \Big[2id+ix_1\cdot\partial_1+ix_2\cdot\partial_2\Big]\langle\Omega \mid T\Big[\phi(x_1)\phi(x_2)\Big] \mid \Omega\rangle \tag{3.10}$$

as result of the terms containing the delta functions cancelling among themselves. Identifying the position space two-point function

$$\tilde{G}^{(2)}(x_1,x_2) = \langle\Omega \mid T\Big[\phi(x_1)\phi(x_2)\Big] \mid \Omega\rangle \tag{3.11}$$

we have derived

$$\tilde{M}^{(2)}(x_1,x_2) \quad = \quad \Big[2id + ix_1\cdot\partial_1 + ix_2\cdot\partial_2\Big]\tilde{G}^{(2)}(x_1,x_2). \tag{3.12}$$

Generalizing to n variables we find

$$\tilde{M}^{(n)}(x_1,x_2,\ldots,x_n) \quad = \quad \Big[nid + ix_1\cdot\partial_1 + ix_2\cdot\partial_2 + \ldots + ix_n\cdot\partial_n\Big]\tilde{G}^{(n)}(x_1,x_2,\ldots,x_n). \tag{3.13}$$

This the naive Ward Identity associated with broken scale invariance. If scale invariance is exact, $S(y) = 0$ and hence $\tilde{M}^{(n)}$ vanishes. Thus, for exact scale invariance we have

$$\Big[nid + ix_1\cdot\partial_1 + ix_2\cdot\partial_2 + \ldots + ix_n\cdot\partial_n\Big]\tilde{G}^{(n)}(x_1,x_2,\ldots,x_n) = 0. \tag{3.14}$$

3.2 Naive Ward Identity in momentum space

Let us look at the Ward Identity of broken scale invariance in momentum space.
The momentum space Green's function is defined by

$$\begin{aligned}(2\pi)^4\ \delta^4(p_1+p_2+\ldots+p_n)\ G^{(n)}(p_1,p_2,\ldots,p_{n-1}) = \\ \int \Pi_i^n\ d^4x_i\ e^{-ip_i\cdot x_i}\ \tilde{G}^{(n)}(x_1,x_2,\ldots,x_n).\end{aligned} \tag{3.15}$$

Restrict to $n = 3$.

$$\begin{aligned}(2\pi)^4\ \delta^4(p_1+p_2+p_3)\ G^{(3)}(p_1,p_2) \quad &= \quad \int d^4x_1\ d^4x_2\ d^4x_3\ e^{-ip_1\cdot x_1}\ e^{-ip_2\cdot x_2}e^{-ip_3\cdot x_3}\tilde{G}^{(3)}(x_1,x_2,x_3). \\ &= \quad \int d^4x_1\ d^4x_2\ d^4x_3\ e^{-ip_1\cdot(x_1-x_3)}\ e^{-ip_2\cdot(x_2-x_3)}\tilde{G}^{(3)}(x_1,x_2,x_3).\end{aligned} \tag{3.16}$$

From

$$\tilde{M}^{(3)}(x_1,x_2,x_3) \quad = \quad i\Big[3d + x_1\cdot\partial_1 + x_2\cdot\partial_2 + x_3\cdot\partial_3\Big]\tilde{G}^{(3)}(x_1,x_2,x_3) \tag{3.17}$$

$$\begin{aligned}\int d^4p_1 \int d^4p_2 \int d^4p_3 (2\pi)^4\ \delta^4(p_1+p_2+p_3)\ \ e^{ip_1\cdot x_1}\ e^{ip_2\cdot x_2}e^{ip_3\cdot x_3}\ M^{(3)}(p_1,p_2,p_3) = \\ i[3d + x_1\cdot\partial_1 + x_2\cdot\partial_2 + x_3\cdot\partial_3\Big] \\ \int d^4p_1 \int d^4p_2 \int d^4p_3 e^{ip_1\cdot x_1}\ e^{ip_2\cdot x_2}\ e^{ip_3\cdot x_3}\ (2\pi)^4\ \delta^4(p_1+p_2+p_3)\ G^{(3)}(p_1,p_2,p_3).\end{aligned} \tag{3.18}$$

i.e.,

$$\begin{aligned}\int d^4p_1 \int d^4p_2\ \ e^{ip_1\cdot(x_1-x_3)}\ e^{ip_2\cdot(x_2-x_3)}\ M^{(3)}(p_1,p_2) = \\ i[3d + x_1\cdot\partial_1 + x_2\cdot\partial_2 + x_3\cdot\partial_3\Big] \int d^4p_1 \int d^4p_2\ e^{ip_1\cdot(x_1-x_3)}\ e^{ip_2\cdot(x_2-x_3)}\ G^{(3)}(p_1,p_2).\end{aligned} \tag{3.19}$$

Using

$$\Big[x_1 \cdot \partial_1 + x_2 \cdot \partial_2 + x_3 \cdot \partial_3\Big]\; e^{-ip_1\cdot(x_1-x_2)}\; e^{-ip_2\cdot(x_3-x_2)} \;=\; \Big[p_1 \cdot \frac{\partial}{\partial p_1} + p_2 \cdot \frac{\partial}{\partial p_2}\Big]\; e^{-ip_1\cdot(x_1-x_2)} \; e^{i-p_2\cdot(x_3-x_2)} \tag{3.20}$$

and doing a partial integration and ignoring surface terms we arrive at

$$M^{(3)}(p_1,p_2) = i\Big[3d - 8 - p_1 \cdot \frac{\partial}{\partial p_1} - p_2 \cdot \frac{\partial}{\partial p_2}\Big] G^{(3)}(p_1,p_2). \tag{3.21}$$

Generalizing to n variables we get the momentum space version of the *naive* Ward identity of broken scale invariance

$$M^{(n)}(p_1,p_2,\ldots,p_{n-1}) = i\Big[nd - 4(n-1) - \sum_{r=1}^{n-1} p_r \cdot \frac{\partial}{\partial p_r}\Big] G^{(n)}(p_1,p_2,\ldots,p_{n-1}). \tag{3.22}$$

3.3 Naive Renormalization Group equation

To convert the above equation to the form of a Renormalization Group equation we do a dimensional analysis. For simplicity we consider the case where the theory has only one dimension full parameter with the dimension of mass, μ. In the bare theory μ may be the bare mass and in the renormalized theory with on-shell renormalization, μ may denote the renormalized mass. Let us recall the definition of the momentum space Green's function:

$$(2\pi)^4\; \delta^4(p_1 + p_2 + \ldots + p_n)\; G^{(n)}(p_1,p_2,\ldots,p_{n-1}) = \int \Pi_i^n\; d^4x_i\; e^{-ip_i\cdot x_i}\; \tilde{G}^{(n)}(x_1,x_2,\ldots,x_n). \tag{3.23}$$

From the dimensional analysis

$$\tilde{G}^n \sim \mu^n. \tag{3.24}$$

Refering back to Eq. (3.15), since $\delta^4(p_1 + p_2 + \ldots + p_n)$ has dimension μ^{-4} and $\Pi_i^n\; d^4x_i$ has dimension μ^{-4n}, we find

$$G^{(n)}(p_1,p_2,\ldots,p_{n-1}) \sim \mu^{4-3n}. \tag{3.25}$$

Thus in terms of a dimensionless function Φ we can write

$$G^{(n)}(p_1,p_2,\ldots,p_{n-1}) = \mu^{4-3n}\; \Phi(\frac{p_1}{\mu},\frac{p_2}{\mu},\ldots,\frac{p_{n-1}}{\mu}). \tag{3.26}$$

We have,

$$\sum_{r=1}^{n-1} p_r \cdot \frac{\partial}{\partial p_r}\Big] G^{(n)}(p_1,p_2,\ldots,p_{n-1}) = \Big[4 - 3n - \mu\frac{\partial}{\partial \mu}\Big] G^{(n)}(p_1,p_2,\ldots,p_{n-1}). \tag{3.27}$$

Substituting back, we get

$$\left[\mu\frac{\partial}{\partial\mu}+n(d-1)\right]G^{(n)}(p_1,p_2,\ldots,p_{n-1})=-iM^{(n)}(p_1,p_2,\ldots,p_{n-1}). \tag{3.28}$$

Consider the case of scalar field theory. Canonically $d=1$. Since in a canonically scale invariant theory $M^{(n)}=0$, we get $\mu\frac{\partial}{\partial\mu}G^{(n)}(p_1,p_2,\ldots,p_{n-1})=0$ which trivially holds (naively speaking, of course).

3.4 Why the naive Ward Identity is too *naive*

Consider the case of exact scale invariance i.e., $M^{(2)}=0$ for $n=2$ in Eq. (3.22),

$$\left[p\cdot\frac{\partial}{\partial p}+4-2d\right]G^{(2)}(p)=0. \tag{3.29}$$

Since $G^{(2)}(p)$ depends only on p^2, we get

$$\left[p^2\frac{\partial}{\partial p^2}+2-d\right]G^{(2)}(p^2)=0. \tag{3.30}$$

The solution is

$$G^{(2)}(p^2)=C\,\frac{1}{p^2}\,(p^2)^{d-1} \tag{3.31}$$

where C is a constant. For $d=1$, the canonical value, the Ward Identity predicts that the momentum dependence of the exact two-point function in the interacting theory is the same as that of the free one. This turns out to be completely wrong! In the next chapter we elaborate on this issue.

CHAPTER 4

Anomalous Dimension in Quantum Field Theory

In the last chapter, we stated that the prediction of the Ward identity of broken scale invariance derived using standard manipulations for the momentum dependence of Greens functions is completely wrong. Where did one go wrong? At the canonical level, in 3+1 dimensions, the scale dimension of scalar field d=1. Anticipating future, we kept it as d. We have found that what appears naturally in the Ward identity is $d-1$. Is there a possibility that d deviates from 1 so that one has $d = 1+\gamma$? The deviation γ is called anomalous dimension since, in the presence of γ in the Ward Identity, one can still recover scale invariance though with non-canonical scale dimension. We have already seen this possibility in our quantum mechanical exercise with $\frac{1}{r^2}$ potential.

4.1 Non-perturbative anomalous dimension

The existence of non-perturbative anomalous dimension in quantum field theory was first demonstrated by K. G. Wilson [27] using the exact solution of massless Thirring model by K. Johnson [28].

The massless Thirring model is the model in 1+1 dimensions of massless fermions interacting via a current-current interaction.

The Lagrangian density is

$$\mathscr{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{\lambda}{2}\bar{\psi}\gamma^\mu \psi \bar{\psi}\gamma_\mu \psi. \tag{4.1}$$

In 1+1 dimensions, canonically, ψ has scale dimension $\frac{1}{2}$ and the coupling λ is dimensionless. Thus the model is exactly scale invariant at the classical level.

Johnson showed that the two-point and four-point Greens functions of the model can be calculated exactly using the facts that (1) vector current j^μ and the axial vector current j_5^μ are conserved exactly and (2) the axial vector current j_5^μ is related to the vector current j^μ in 1+1 dimensions by $j_5^\mu = \varepsilon^{\mu\nu}\, j_\nu$ in 1+1 dimensions with $\varepsilon^{\mu\nu}$ being the antisymmetric tensor.

The two point function is given by

$$\begin{aligned} G(x-y) &= i\,\langle\Omega \mid T(\psi(x)\bar{\psi}(y)) \mid \Omega\rangle \\ &= exp\Big\{-i\lambda\,(a-\bar{a})\, D_0(x-y)\Big\}\, G_0(x-y) \end{aligned} \tag{4.2}$$

with $a = \frac{1}{1-\lambda/(2\pi)}$ and $\bar{a} = \frac{1}{1+\lambda/(2\pi)}$. Here $\mid\Omega\rangle$ is the vacuum state, T is the time-ordering symbol, $G_0(x-y)$ is the free massless Dirac propagator and $D_0(x-y)$ is the free propagator of a massless scalar field. Explicitly

$$G_0(x-y) = \frac{1}{2\pi}\gamma^\mu (x-y)_\mu\, \frac{1}{(x-y)^2 + i\eta} \tag{4.3}$$

and

$$D_0(x-y) = -\frac{i}{4\pi} \ln\,[\frac{-(x-y)^2 + i\eta}{x_0^2}]. \tag{4.4}$$

To keep dimensions correct, an arbitrary constant length x_0 is introduced which remains unchanged under scale transformation.

Under scale transformation where the coordinate $x \to sx$

$$\psi(x) \to U^\dagger(s)\psi(x)U(s) = s^d\psi(sx). \tag{4.5}$$

Demanding the canonical commutation relation

$$\{\psi(x), \psi^\dagger(y)\} = i\delta(x-y) \tag{4.6}$$

to be invariant under scale transformation, we get $d = \frac{1}{2}$.

Consider the exact two-point function

$$\begin{aligned} G(x-y) &= i\,\langle\Omega \mid T(\psi(x)\bar{\psi}(y)) \mid \Omega\rangle \\ &= i\,\langle\Omega \mid U(s)T(U^\dagger(s)\psi(x)U(s)U^\dagger(s)\bar{\psi}(y))U(s)U^\dagger(s) \mid \Omega\rangle \\ &= s^{2d}\,G(sx-sy) \end{aligned} \tag{4.7}$$

using the invariance of the vacuum. (Exact scale invariance of the theory at the tree level is preserved in the exact solution since no new parameter with dimension of mass is present.) Thus we have

$$s^{2d}\,exp\Big\{-i\lambda(a-\bar{a})\,D_0(sx-sy)\Big\}\,G_0(sx-sy) = exp\Big\{-i\lambda(a-\bar{a})\,D_0(x-y)\Big\}\,G_0(x-y). \tag{4.8}$$

Using the explicit expressions for G_0 and D_0 given in Eq. (4.3) and Eq. (4.4) respectively, we arrive at

$$d = \frac{1}{2} + \frac{\lambda^2/(4\pi^2)}{1-\lambda^2/(4\pi^2)} = \frac{1}{2} + \gamma. \tag{4.9}$$

The anomalous dimension γ can even become ∞ as the coupling $\lambda \to \pm 2\pi$.

As a side comment we note that the beta function β is exactly zero in this model. This arises from that fact that the interaction is of the form vector current - vector current, and the current is conserved.

We have to wonder about the source of the anomalous dimension. In perturbation theory at least, Thirring model has ultraviolet divergences. To make sense of the theory we need put an ultraviolet cutoff Λ. As we try to take Λ to infinity, the limit is not smooth. In other words, the short distance limit is not that of the free field theory. Anomalous dimension indeed is the mark of the microscopic length scale $\frac{1}{\Lambda}$ that is left behind as the cutoff is taken to infinity and renormalization is performed by sweeping the trouble under the rug by means of wave function renormalization. In this sense the anomalous dimension is a cousin of the grin of the famous *Cheshire-Cat* in Alice's Adventures in Wonderland. Cat (cutoff) has disappeared leaving only the grin (anomalous dimension)! In the next section we elaborate on this using the computation of anomalous dimension in perturbation theory both in the cutoff and renormalized versions.

4.2 Example of anomalous dimension in perturbation theory

Consider photon propagator in QED in the Feynman gauge $\lambda = 1$. At tree level

$$iD^{\mu\nu}_{(0)}(k) = i\,(-)\frac{g^{\mu\nu}}{k^2 + i\eta}. \tag{4.10}$$

To second order in the coupling we get,

$$i\,D^{\mu\nu}_{(2)}(k) = i\,D^{\mu\nu}_{(0)}(k) + i\,D^{\mu\rho}_{(0)}(k)\;i\,\Pi_{\rho\sigma}(k)\;i\,D^{\sigma\nu}_{(0)}(k) \tag{4.11}$$

where the photon self energy or the vacuum polarization is given by

$$i\,\Pi^{\mu\nu}(k) = e^2 \int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\gamma^\mu \frac{1}{\gamma^\rho p_\rho - m + i\eta}\gamma^\nu \frac{1}{\gamma^\rho p_\rho - k_\rho - m + i\eta}\right]. \tag{4.12}$$

With $\Pi_{\mu\nu}(k) = [k_\mu k_\nu - g_{\mu\nu}k^2]\,\Pi(k^2)$, to order e^2, we have

$$D^{\mu\nu}_{(2)}(k) = -\frac{g^{\mu\nu}}{k^2 + i\eta}\,\left[1 - \Pi(k^2)\right] + \ldots \tag{4.13}$$

where $\ldots$ represents uninteresting (in the present context!) terms containing k^μ, k^ν, etc.

In Pauli-Villars regularization, we get

$$1 - \Pi(k^2) = 1 - \frac{\alpha}{3\pi}\,ln\frac{M^2}{m^2} + 2\frac{\alpha}{\pi}\int_0^1 dz\,z\,(1-z)\,\ln\left[\,\frac{m^2 - k^2 z(1-z)}{m^2}\,\right] \tag{4.14}$$

where M is the Pauli-Villars mass ($= \Lambda$). In the on-shell renormalization scheme,

$$\Pi(k^2) = \Pi(k^2)_{div} + \Pi(k^2)_{fin} \tag{4.15}$$

with $\Pi(k^2)_{div} = \Pi(k^2 = 0)$. Then the photon propagator to order e^2 is given by

$$D^{\mu\nu}_{(2)}(k) = Z_3(\Lambda)\,D^{\mu\nu}_{(R)}(k). \tag{4.16}$$

The renormalization constant

$$Z_3(\Lambda) = 1 - \frac{\alpha}{3\pi}\,\ln\left[\frac{\Lambda^2}{m^2}\right] \tag{4.17}$$

and the renormalized photon propagator to order α

$$D^{\mu\nu}_{(R)}(k) = -\frac{g^{\mu\nu}}{k^2 + i\eta}\left[1 + 2\frac{\alpha}{\pi}\int_0^1 dz\,z\,(1-z)\,\ln\left[\,\frac{m^2 - k^2 z(1-z)}{m^2}\,\right]\right]. \tag{4.18}$$

Let us study the behaviour of $D^{\mu\nu}_{(R)}(k)$ in the deep Euclidean region ($k^2 \to -k_E^2$, $|\,k_E\,| \to \infty$). In this limit

$$D^{\mu\nu}_{(R)}(k) \approx g^{\mu\nu}\frac{1}{k^2}\left[1 + \frac{e^2}{12\pi^2}\,\ln\frac{k^2}{m^2}\right]. \tag{4.19}$$

Thus to order we can write (in the deep Euclidean region)

$$D^{\mu\nu}_{(R)}(k) = g^{\mu\nu}\frac{1}{k^2}\left(\frac{k^2}{m^2}\right)^{(e^2/(12\pi^2))} . \tag{4.20}$$

Recalling the prediction of the naive Ward identity for the propagator for asymptotic momenta, we identify the perturbative anomalous dimension to order e^2 to be $\gamma = \frac{e^2}{12\pi^2}$.

Note: We have pretended that $1+x \approx e^x$. Recalling $e^x = 1+x+\frac{x^2}{2!}+\ldots$ and $1/(1-x) = 1+x+x^2+\ldots$ we can't be sure whether the identification will hold to higher order in e^2. In fact, *it doesn't*. Thus the anomalous dimension in the present example is more like a would-be anomalous dimension.

So far we identified the anomalous dimension in the renormalized theory. Next we proceed to calculate the same in the bare cutoff theory.

The definition of the anomalous dimension in terms of the wave function renormalization constant for the field ϕ is

$$\begin{aligned}\gamma_\phi &= \mu^2\frac{\partial}{\partial\mu^2}\ln Z_\phi \mid_{(\Lambda,\lambda_0)} \\ &= -\frac{\partial}{\partial ln\Lambda^2}\ln Z_\phi \mid_{(\mu,\lambda_0)}\end{aligned} \tag{4.21}$$

where λ_0 is the bare coupling constant and μ is the renormalization scale. Using

$$Z_3(\Lambda) = 1 - \frac{\alpha}{3\pi}\ln\left[\frac{\Lambda^2}{m^2}\right] \tag{4.22}$$

we get $\gamma = \frac{e^2}{12\pi^2}$ which agrees with the value calculated in the renormalized theory.

4.3 Story so far

Allowing for an arbitrary scale dimension d for the field, our naive Ward Identity of broken scale invariance for the n-point momentum space Greens function took the form

$$\left[\mu\frac{\partial}{\partial\mu} + n(d-1)\right]G^{(n)}(p_1,p_2,\ldots,p_{n-1}) = -iM^{(n)}(p_1,p_2,\ldots,p_{n-1}). \tag{4.23}$$

Now we have found enough motivation to modify this equation to read

$$\left[\mu\frac{\partial}{\partial\mu} + n\gamma\right]G^{(n)}(p_1,p_2,\ldots,p_{n-1}) = -iM^{(n)}(p_1,p_2,\ldots,p_{n-1}). \tag{4.24}$$

Thus, as a result of short distance divergences in the wave function renormalization constant, the scale dimension becomes interaction dependent but scaling is still recovered in the asymptotic momenta limit.

Question arises whether this is all. A resounding answer turns out to be NO. The next (and final) correction to the naive Ward Identity introduces the concepts of effective coupling constant and another renormalization function named beta function. We take this up in the next chapter.

CHAPTER 5

Running Coupling Constant and the β Function

In this chapter we verbatim follow (for the discussion of ϕ^4 theory) Wilson's treatment in Ref. ([29]) which provides the most intuitive picture of renormalization as a problem of infinitely many scales (popularly known under the misnomer *Wilson Renormalization Group*. In this work, Wilson starts with a generic situation. In general, in nature, a physical system has a number of length scales each of which is described by a *different* set of laws. Wilson gives the example of water which on a macroscopic scale with wavelength of the order of meters, is described by an equation which is governed by the laws of hydrodynamics. Some parameters appear in this equation, namely, density and viscosity. To *determine* these parameters, one has to solve the problem of water on a smaller length scale, the atomic scale. At this scale, the relevant equation (Schroedinger equation) is governed by a different physical law, namely, quantum mechanics. At the atomic scale, we need parameters like masses of nuclei which are just parameters at this scale. Thus at each length scale we need a few parameters to solve the problem. Important observation of Wilson is that *the parameters for a given length scale are determined from the parameters for the next smaller length scale*. Thus in a typical physical system encountered in nature, different length scales are governed by different laws of physics.

Now come to quantum field theory. Take the simplest field theory, $\frac{g_0}{4!}\phi^4$ as an example. Consider the 4-point function.

We ignore the t-channel and u-channel contributions at the moment. They do not affect the discussion in the following in a significant way. The tree level contributes $-i\ g_0$. Second order contributes $-g_0^2 c_1 \int d^4k\ \frac{1}{k^2+m^2}\ \frac{1}{(k+q)^2+m^2}$. We are in Euclidean space. The coefficient $c_1 > 0$. Integral contains a logarithmic divergence coming from the region of large k where the magnitude of k is much larger than the magnitude of q or m. Henceforth we denote the magnitudes by k and q. Since the external momentum q is at our disposal, we consider $q >> m$. Then the log divergent part of the integral is $\int_q^\infty \frac{d^4k}{k^4} = \int_q^\infty \frac{dk}{k}$. Let us divide the interval from q to ∞ in the following way:

$$\int_q^\infty \frac{dk}{k} = \int_q^{2q} \frac{dk}{k} + \int_{2q}^{4q} \frac{dk}{k} + \dots . \tag{5.1}$$

Note that the contribution from each subinterval is *finite*. Divergence is due to the presence of an infinite number of subintervals from q to ∞. Each subinterval represents the contribution of different momentum scale. Divergence arises because every momentum scale from q to ∞ is trying to make an equal contribution; and there are an infinite number of momentum scales.

It is obvious that an infinite number of energy scales all contributing equal finite amount is specific to a logarithmically diverging integral. For a linearly or quadratically divergent integral, higher energy scales will contribute more and for a convergent integral higher energy scales will contribute less. So why make a big fuss? The answer is that the holy grail of finding the continuum limit of a quantum field theory depends on the β function and the beta function arises from logarithms.

A major difference between the example of water and ϕ^4 theory is to be noted. For water, different length scales are governed by different physical laws. But we see shortly that for ϕ^4 theory, physical laws describing different length scales do not change from scale to scale. However, the parameters appearing in ϕ^4 theory do change from scale to scale. We now try to see how this comes about. Anticipating future, we will see that instead of a single coupling constant g_0, there will be a momentum dependent coupling constant $g_{eff}(q)$.

5.1 In ϕ^4 theory

From the explicit calculation, (including all the relevant graphs) we define

$$-ig_{eff}(q) = -g_0 + ic_2 g_0^2 \int_q^\infty \frac{dk}{k}\ , \quad c_2 > 0, \tag{5.2}$$

or

$$g_{eff}(q) = g_0 + c g_0^2 \int_q^\infty \frac{dk}{k}\ , \quad c < 0. \tag{5.3}$$

With analogy of the water example, we expect to determine $g_{eff}(q)$ knowing $g_{eff}(2q)$. Indeed, from our definitions,

$$g_{eff}(q) = g_0 + c g_0^2 \int_q^{2q} \frac{dk}{k} + c g_0^2 \int_{2q}^\infty \frac{dk}{k}. \tag{5.4}$$

Thus

$$g_{eff}(q) = g_{eff}(2q) + c g_0^2 \int_q^{2q} \frac{dk}{k}. \tag{5.5}$$

This equation still contains g_0^2. Note that since we stopped the perturbative calculation to order g_o^2, the above equation is not correct to order g_0^3 and higher, so within the accuracy of the approximation, it is legitimate to replace g_0^2 by $g_{eff}^2(2q)$ in the above equation. Thus we have the remarkable result

$$g_{eff}(q) = g_{eff}(2q) + c\ g_{eff}^2(2q)\ \ln 2 \qquad c < 0 \tag{5.6}$$

As expected, when $g_{eff}(q)$ is expressed in terms of $g_{eff}(2q)$, only one momentum scale is involved, that from q to $2q$. Thus we no longer see any divergence.

From Eq. (5.6), we find that, since $c < 0$, $g_{eff}(2q) > g_{eff}(q)$, i.e., effective coupling increases as momentum scale increases.

From what we got, we can arrive at a differential equation for the effective coupling.

We have

$$g_{eff}(q-\delta q)=g_{eff}(q)+cg_{eff}^2(q)\int_{q-\delta q}^{q}\frac{dk}{k}. \tag{5.7}$$

Thus

$$g_{eff}(q)-g_{eff}(q-\delta q)=-c\ g_{eff}^2(q)\ [\ln q-\ \ln\ (q-\delta q)], \tag{5.8}$$

or

$$dg_{eff}(q)=-cg_{eff}^2(q)d\ \ln q. \tag{5.9}$$

From Eq. (5.9), we have

$$\frac{dg_{eff}(q)}{d\ \ln q}=-cg_{eff}^2(q)=\beta(g_{eff}) \tag{5.10}$$

which defines the β function of renormalization group.

Integrating,

$$g_{eff}(q)=\frac{g_{eff}(q_0)}{1+cg_{eff}(q_0)\ \ln\ (q/q_0)},\quad c<0 \tag{5.11}$$

For ϕ^4 theory, near small g, we have found that $\beta(g)>0$, since $c<0$.

5.2 In QED

5.2.1 β function

In QED, the relation between the renormalized charge e_R and bare charge e_0 is given by $e_R = e_0\,\overline{Z}$ with $\overline{Z}=\sqrt{Z_3}$, Z_3 being the photon renormalization constant. β function is defined by (see, for example, Cheng and Li [7], chapter 3)

$$\begin{aligned}
\beta &= 2\mu^2\frac{\partial}{\partial\mu^2}\,e_R(e_0,\Lambda/\mu)\,|_{(e_0,\Lambda)}\\
&= e_R\,\mu\frac{\partial}{\partial\mu}\ln\overline{Z}(e_0,\Lambda/\mu)\,|_{(e_0,\Lambda)}\quad \text{using}\ \ e_R=e_0\,\overline{Z}\\
&= -e_R\,\frac{\partial}{\partial\ \ln\Lambda}\ln\overline{Z}(e_0,\Lambda/\mu)\,|_{(e_0,\mu)}\\
&= -\,e_R\,\frac{\partial}{\partial\ \ln\Lambda^2}\ln Z_3\,|_{(e_R,\mu)}\\
&= \frac{e_R^3}{12\pi^2}
\end{aligned} \tag{5.12}$$

using the explicit result from Pauli-Villars regularization,

$$Z_3(\Lambda)=1-\frac{e_R^2}{12\pi^2}\ \ln\ \frac{\Lambda^2}{m^2}. \tag{5.13}$$

Thus the positivity of the β function in QED is directly connected with $Z_3 < 1$ which is a consequence of the positivity of the spectral function. The reason for this connection is of course the Ward Identity $Z_1 = Z_2$.

We have

$$\frac{de_{eff}(q)}{d\ \ln q} = \beta(e_{eff}) = \frac{e_{eff}^3}{12\pi^2}. \tag{5.14}$$

Integrating

$$\alpha_{eff}(q) = \frac{\alpha_{eff}(q_0)}{1 - \frac{\alpha_{eff}(q_0)}{3\pi}\ \ln \frac{q^2}{q_0^2}}. \tag{5.15}$$

As a consequence of positive β, effective coupling in QED increases with momentum and eventually blows up at $q^2 = q_0^2 \times 10^{560}$. Of course long before this happens the lowest order result which is used to arrive at this conclusion becomes invalid. This is because the effective coupling is growing with momentum and neglected terms in the perturbation expansion become more important than those that are kept. The pole in the effective fine structure constant is called a Landau ghost. Why is it called a ghost? To answer this we look at the photon propagator in the same approximation.

5.2.2 Photon propagator in the leading logarithmic approximation

Recall the form of the renormalized (on-shell scheme) photon propagator to order α in QED.

$$D^{\mu\nu}_{(R)}(k) = -\frac{g^{\mu\nu}}{k^2 + i\eta}\left[1 + 2\frac{\alpha}{\pi}\int_0^1 dz\, z\,(1-z)\ \ln\left[\frac{m^2 - k^2 z(1-z)}{m^2}\right]\right]. \tag{5.16}$$

By summing the leading logarithms, we get

$$D^{\mu\nu}_{(R)}(k) = -\frac{g^{\mu\nu}}{k^2 + i\eta}\frac{1}{1 - 2\frac{\alpha}{\pi}\int_0^1 dz\, z\,(1-z)\ \ln\left[\frac{m^2 - k^2 z(1-z)}{m^2}\right]}. \tag{5.17}$$

Note that this approximation is also called bubble approximation [30, 31].

In the deep Euclidean region, $k^2 \to -k_E^2$, $\mid k_E \mid \to \infty$,

$$D^{\mu\nu}_{(R)}(k) \approx \frac{g^{\mu\nu}}{k_E^2}\frac{1}{1 - \alpha/(3\pi)\ \ln \frac{k_E^2}{m^2}}. \tag{5.18}$$

Thus the photon propagator develops an extra pole at $k_E^2 = m^2 \times 10^{560}$.

(To match with the result from the previous subsection, we note that for $q_0 = m$, the electron mass, $\alpha_{eff}(q_0) = 1/137$.)

Thus we have found that in Minkowski space, photon propagator has, in addition to the physical pole at $k^2 = 0$, an extra pole at space-like momentum $k^2 = -m^2 \times 10^{560}$. Physical particles should correspond to poles of the propagator for time-like momentum so a pole of the propagator for space-like momentum has to be a ghost!

This indicates a possible inconsistency of QED at large (indeed VERY, VERY LARGE) energy scales. Can one trust this conclusion from partial summation of perturbation series? We note that the problem arose from the positivity of the beta function (at least for small coupling) which resulted from the photon wave function renormalization constant Z_3 being less than one. But this constraint is outside of perturbation theory, being a consequence of positivity constraint on the spectral function. The result of the inconsistency of QED at high energies even though derived in perturbation theory may be valid outside of perturbation theory.

Everything is not lost, however. QED is supposed to be the physical law for atomic systems where the typical energy scale is of the order of electron volts. At this energy scale, the rest mass of the electron itself, being 10^6 bigger, provides a natural high momentum cutoff!

CHAPTER 6

Renormalization Group Equations

In this chapter first we look at the modified Ward identity of broken scale invariance (popularly known as renormalization group equation) in the context of on-shell renormalization. This is the well-known Callan-Symanzik equation. Then we illustrate the arbitrariness in the renormalization process with the example of one loop vacuum polarization in QED in the contexts of Pauli-Villars regularization and dimensional regularization. Then we look at renormalization group equations in some subtraction schemes other than on-shell renormalization. Lastly we take up the question of *group* in renormalization group.

6.1 Callan-Symanzik Equation

In chapter 3, we started with the study of the *naive* Ward identity of broken scale invariance. At the end of chapter 4 we found (hopefully) convincing evidence for the deviation of scale dimension from its canonical (tree level) value. The reason for this deviation is the divergence of wave function renormalization constants as the ultra-violet cutoff becomes very large. This divergence results in *finite* anomalous dimensions in the renormalized theory which signals deviation from canonical scaling laws. Thus we arrived at

$$\left[\mu\frac{\partial}{\partial\mu}+n\gamma\right]G^{(n)}(p_1,p_2,\ldots,p_{n-1})=-iM^{(n)}(p_1,p_2,\ldots,p_{n-1}). \tag{6.1}$$

For one particle irreducible, truncated vertex function $\Gamma^{(n)}$ the corresponding equation is

$$\left[\mu\frac{\partial}{\partial\mu}-n\gamma\right]\Gamma^{(n)}(p_1,p_2,\ldots,p_{n-1})=-i\overline{M}^{(n)}(p_1,p_2,\ldots,p_{n-1}) \tag{6.2}$$

where $\overline{M}^{(n)}$ denotes the corresponding one particle irreducible, truncated vertex function with the insertion of the integral of the divergence of scale current. Note the change in sign of the term containing γ in going from Eq. (6.1) to Eq. (6.2). Note that in the absence of the explicit scale breaking (right hand side of above equations zero), the Greens functions will obey scaling laws with naive scale dimension modified by anomalous dimension.

But in chapter 5, we encountered a new aspect of divergence apart from modifications to field normalizations. We encountered the phenomena of coupling constant changing with scale. This was controlled by the β function. Thus we need to modify the naive Ward identity further, to take in to account the variation of coupling constant with scale. Accordingly we add a term proportional to $\frac{\partial}{\partial g}$ to the right hand side. (This effect survives even when the right hand side vanishes, i.e., when there are no dimension full parameters in the tree level Lagrangian.) The effect can be avoided only if β function is zero everywhere. To reflect this fact, we take the modified term to $\beta\, \frac{\partial}{\partial g}$.

It turns out that in *renormalizable* field theories there are no more modifications (for a simple demonstration, see Ref. [24]). Thus the modified Ward identity of broken scale invariance is

$$\left[\mu \frac{\partial}{\partial \mu} - n\gamma + \beta\, \frac{\partial}{\partial g}\right] \Gamma^{(n)}(p_1, p_2, \ldots, p_{n-1}) = -i\overline{M}^{(n)}(p_1, p_2, \ldots, p_{n-1}). \tag{6.3}$$

One basic ingredient in the beginning of our *derivation* of Eq. (6.3) was that the theory is characterized by a single mass. This is true only in cutoff regularization with on-shell subtraction. In this scheme, since there is only one mass parameter, namely renormalized mass m_R which is the same as the physical mass, the dimensionless functions γ and β can depend only on the renormalized dimensionless coupling constant. In the renormalized theory, the Greens function with the insertion, $\overline{M}^{(n)}$ can acquire extra renormalization. We account for this fact by multiplying $\overline{M}^{(n)}$ by a factor α. Thus we *heuristically* arrive at

$$\left[m_R \frac{\partial}{\partial m_R} - n\gamma(g_R) + \beta(g_R)\, \frac{\partial}{\partial g_R}\right] \Gamma^{(n)}(p_1, p_2, \ldots, p_{n-1}) = -i\; \alpha\, \overline{M}^{(n)}(p_1, p_2, \ldots, p_{n-1}). \tag{6.4}$$

which is nothing but the Callan-Symanzik equation of renormalization group. We remind the reader that this equation is associated with the scheme of on-shell subtraction.

6.2 Arbitrariness in the process of renormalization

We look at the arbitrariness in the process of renormalization with example of one loop vacuum polarization in QED.

6.2.1 Pauli-Villars Regularization

First let us consider Pauli-Villars regularization.

Our starting point is the Pauli-Villars regulated expression for the vacuum polarization

$$1 - \Pi(k^2) = 1 - \frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2} + 2\frac{\alpha}{\pi} \int_0^1 dz\, z(1-z) \ln \frac{m^2 - k^2(z(1-z)}{m^2}. \tag{6.5}$$

To isolates the divergence we write

$$\Pi(k^2) = \Pi(k^2)_{div} + \Pi(k^2)_{fin} \tag{6.6}$$

where the meanings of the subscripts are self-evident.

On-shell subtraction

In the on-shell subtraction scheme we choose

$$\Pi(k^2)_{div} = \Pi(k^2 = 0) = \frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{m^2}. \tag{6.7}$$

Then

$$1 - \Pi(k^2)_{fin} = 1 + 2\frac{\alpha}{\pi} \int_0^1 dz\, z(1-z) \ln \frac{m^2 - k^2(z(1-z)}{m^2}. \tag{6.8}$$

As long as a non-zero external momentum k flows into the loop, we expect it to provide the infra-red regulator and we expect no singularities in the vacuum polarization in the limit of electron mass m tending to zero. This is evident in the regulated expression Eq. (6.5) since, as $m \to 0$

$$1 - \Pi(k^2) \to 1 + 2\frac{\alpha}{\pi} \int_0^1 dz\, z(1-z) \ln \frac{-k^2 z(1-z)}{\Lambda^2}. \tag{6.9}$$

However the finite part of $\Pi(k^2)$ given in Eq. (6.8) is infra-red divergent in the limit $m \to 0$. Noting that, in this scheme the infinite part given in Eq. (6.7) is also infra-red divergent in this limit, it is clear that the infra-red divergence is produced solely by the on-shell renormalization scheme. This gives rise to hope that one may avoid it by choosing another subtraction scheme. This was noticed by Gell-Mann and Low [9] who proposed the off-shell subtraction as the remedy.

Off-shell subtraction

Define

$$\Pi(k^2) = \Pi(k^2 = -\lambda^2), \;\; \lambda^2 > 0. \tag{6.10}$$

Then

$$\begin{aligned} 1 - \Pi(k^2)_{div} &= 1 - 2\frac{\alpha}{\pi} \int_0^1 dz\, z(1-z) \ln \frac{\Lambda^2}{m^2 + \lambda^2 z(1-z)} \\ &= -\frac{\alpha}{3\pi} \ln \frac{\Lambda^2}{\lambda^2}, \;\; \text{for } \lambda^2 >> m^2 \, . \end{aligned} \tag{6.11}$$

Finite part is

$$\begin{aligned} 1 - \Pi(k^2)_{fin} &= 1 + 2\frac{\alpha}{\pi} \int_0^1 dz\, z(1-z) \ln \frac{m^2 - k^2 z(1-z)}{m^2 + \lambda^2 z(1-z)} \\ &\to 1 + \frac{\alpha}{3\pi} \ln \frac{-k^2}{\lambda^2}, \;\; \text{for } -k^2, \lambda^2 >> m^2. \end{aligned} \tag{6.12}$$

Thus both the divergent and finite parts of $\Pi(k^2)$ are free of infra-red divergence in the limit $m \to 0$.

Not that there is nothing special about λ. The value of λ is chosen depending upon the occasion.

6.2.2 Dimensional Regularization

Next we consider dimensional regularization.

In dimensional regularization we get

$$\Pi(k^2) = \frac{e^2}{6\pi^2}\,\frac{1}{\varepsilon} - \frac{e^2}{12\pi^2}\,(\gamma_E - \ln 4\pi) - \frac{e^2}{2\pi^2}\int_0^1 dz\, z(1-z)\,\ln\frac{m^2 - k^2 z(1-z)}{\mu^2} \tag{6.13}$$

where μ is the mass introduced to make e dimensionless. As a result e depends on μ.

On-shell subtraction

$$\Pi(k^2)_{div} = \Pi(k^2=0) = \frac{e^2}{6\pi^2}\,\frac{1}{\varepsilon} - \frac{e^2}{12\pi^2}\,(\gamma_E - \ln 4\pi). \tag{6.14}$$

Then Z_3 depends on on μ, α, m, and ε.

$$\Pi(k^2)_{fin} = -\frac{e^2}{2\pi^2}\int_0^1 dz\, z(1-z)\,\ln\frac{m^2 - k^2 z(1-z)}{\mu^2} \tag{6.15}$$

looks the same as $\Pi(k^2)_{fin}$ in Pauli-Villars regularization scheme.

Off-shell subtraction

$$\begin{aligned}\Pi(k^2)_{div} &= \Pi(k^2 = -\lambda^2) \\ &= \frac{e^2}{6\pi^2}\,\frac{1}{\varepsilon} - \frac{e^2}{12\pi^2}\,(\gamma_E - \ln 4\pi)\frac{e^2}{2\pi^2}\int_0^1 dz\, z(1-z)\,\ln\frac{m^2 + \lambda^2 z(1-z)}{\mu^2}.\end{aligned} \tag{6.16}$$

The finite part is

$$\Pi(k^2)_{fin} = -\frac{e^2}{2\pi^2}\int_0^1 dz\, z(1-z)\,\ln\frac{m^2 - k^2 z(1-z)}{m^2 + \lambda^2 z(1-z)} \tag{6.17}$$

which looks the same as in Pauli-Villars regularization.

Minimal subtraction

In dimensional regularization, one has more choices of subtraction schemes than cut-off regulator schemes. For example, one can choose

$$\Pi(k^2)_{div} = \frac{e^2}{6\pi^2}\,\frac{1}{\varepsilon} \tag{6.18}$$

which is called Minimal Subtraction scheme. In this scheme

$$\Pi(k^2)_{fin} = -\frac{e^2}{12\pi^2}\,(\gamma_E - \ln 4\pi) - \frac{e^2}{2\pi^2}\int_0^1 dz\, z(1-z)\,\ln\frac{m^2 - k^2 z(1-z)}{\mu^2}\,. \tag{6.19}$$

Modified minimal subtraction

Sometimes the choice

$$\Pi(k^2)_{div} = \frac{e^2}{6\pi^2}\,\frac{1}{\varepsilon} - \frac{e^2}{12\pi^2}\,(\gamma_E - \ln 4\pi) \tag{6.20}$$

is made. Then

$$\Pi(k^2)_{fin} = -\frac{e^2}{2\pi^2}\int_0^1 dz\, z(1-z)\,\ln\frac{m^2 - k^2 z(1-z)}{\mu^2}\,. \tag{6.21}$$

6.3 Other Renormalization Group equations

6.3.1 't Hooft-Weinberg Equation

The 't Hooft - Weinberg renormalization group equation is associated with minimal subtraction scheme in the context of dimensional regularization. In this scheme, the bare mass and the bare coupling does not depend on μ, the arbitrary mass scale introduced to keep the dimension of Lagrangian density fixed at d in d space-time dimensions. It follows that the renormalized masses and couplings depend on it. In the minimal subtraction scheme, the renormalization constants do not depend explicitly on μ. Their dependence on μ is entirely through their dependence on g_R which in turn depends on μ. It follows that

$$\frac{d}{d\mu}\,\Gamma^{(n)}(p,g,m) = 0 \tag{6.22}$$

where $\Gamma^{(n)}(p,g,m)$ is the bare, amputed, 1PI Greens function which depends on the bare parameters m and g. But

$$\Gamma^{(n)}(p,g,m) = Z^{-\frac{n}{2}}\,\Gamma_R^{(n)}(p,g_R,m_R,\mu) \tag{6.23}$$

where $\Gamma_R^{(n)}(p,g_R,m_R,\mu)$ is the renormalized amputed, 1PI Greens function. Substituting Eq. (6.23) in to Eq. (6.22), we get the 't Hooft - Weinberg renormalization group equation

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R)\,\frac{\partial}{\partial g_R} - \gamma_m(g_R)\,m_R\frac{\partial}{\partial m_R} - n\gamma(g_R)\right]\Gamma_R^{(n)}(p,g_R,m_R,\mu) = 0. \tag{6.24}$$

Here

$$\begin{aligned}
\beta(g_R) &= \mu\frac{\partial\, g_R}{\partial\mu} \\
\gamma_m(g_R) &= -\frac{\mu}{m_R}\frac{\partial\, m_R}{\partial\mu} \\
\gamma(g_R) &= \frac{\mu}{2Z}\,\frac{\partial Z}{\partial\mu}.
\end{aligned} \tag{6.25}$$

The 't Hooft-Weinberg equation given in Eq. (6.24) shows the invariance of physical observables with respect to the choice of the arbitrary mass scale μ in the minimal subtraction scheme in the context of dimensional regularization.

6.3.2 Gell-Mann - Low Equation

In the Gell-Mann - Low renormalization scheme, mass renormalization is done on-shell but other subtractions are done off-shell. Let μ denote this off-shell scale ($k^2 = -\mu^2$). Thus m_R, the renormalized mass is the physical mass which does not depend on μ. Now, β and γ can depend on g_R and $\frac{m_R}{\mu}$. Thus the Gell-Mann-Low renormalization group equation is

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g_R, \frac{m_R}{\mu})\,\frac{\partial}{\partial g_R} - n\gamma(g_R, \frac{m_R}{\mu})\right]\Gamma_R^{(n)}(p, g_R, m_R, \mu) = 0. \tag{6.26}$$

6.4 Where is the *group* in Renormalization Group?

There is three-fold arbitrariness in the process of renormalization.

- Arbitrariness in the choice of regulator
- Within a regulator, arbitrariness in the choice of subtraction scheme.
- Within a subtraction scheme, arbitrariness in the choice of subtraction point.

In this discussion we follow Wilson [32]. In a particular renormalization scheme, call Γ_μ and Γ_B the renormalized and bare 1PI amputed Greens functions. The subscript μ denotes the subtraction point. We write

$$\Gamma_\mu = \overline{Z}(\mu)\,\Gamma_B \tag{6.27}$$

where $\overline{Z}(\mu)$ denotes the appropriate product of renormalization constants defined within one scheme. If we choose another subtraction point μ',

$$\Gamma_{\mu'} = \overline{Z}(\mu')\,\Gamma_B\ \ . \tag{6.28}$$

Thus

$$\Gamma_{\mu'} = \frac{\overline{Z}(\mu')}{\overline{Z}(\mu)}\Gamma_\mu = Z(\mu',\mu)\,\Gamma_\mu. \tag{6.29}$$

By definition, $Z(\mu',\mu)$ is finite. Since Z is dimensionless, it follows that $Z(\mu',\mu) = Z(\frac{\mu'}{\mu})$. Let us denote $s = \frac{\mu'}{\mu}$.

Let us now consider a set of all possible $Z(s)$ for arbitrary s.

Unit element exists, $Z(s=1) = 1$.

For every element $Z(s)$, there is an inverse $Z^{-1}(s) = Z(s^{-1})$.

Multiplication law: $Z(s')\,Z(s'') = Z(s'\,s'')$ provided $s' = \frac{\mu'}{\mu''}$, $s'' = \frac{\mu''}{\mu'''}$.

Note that the multiplication law does not hold for arbitrary $Z(s')$ and $Z(s'')$.

Let us consider an example explicitly in perturbation theory to order α. We consider the vacuum polarization in QED in Pauli-Villars regularization scheme with off-shell subtraction scheme (subtraction done at $k^2 = -\lambda^2$). Then

$$\overline{Z}(\lambda) = Z_3(\lambda) = 1 - 2\frac{\alpha}{\pi}\int_0^1 dzz(1-z)\ \ln\left[\frac{\Lambda^2}{m^2+\lambda^2 z(1-z)}\right]. \tag{6.30}$$

Then

$$Z\left(\frac{\lambda_2}{\lambda_1}\right) = 1 + 2\frac{\alpha}{\pi}\int_0^1 dz\, z(1-z)\ \ln\left[\frac{m^2+\lambda_2^2 z(1-z)}{m^2+\lambda_1^2 z(1-z)}\right] \tag{6.31}$$

It follows that

$$Z\left(\frac{\lambda_2}{\lambda_1}\right) Z\left(\frac{\lambda_1}{\lambda_3}\right) = Z\left(\frac{\lambda_2}{\lambda_3}\right). \tag{6.32}$$

In the limit of zero electron mass (see Ref. [32]),

$$Z\left(\frac{\lambda_2}{\lambda_1}\right) = 1 + \frac{\alpha}{3\pi}\ln\frac{\lambda_2^2}{\lambda_1^2}. \tag{6.33}$$

Then it follows that for any two elements $Z\left(\frac{\lambda_2}{\lambda_1}\right)$ and $Z\left(\frac{\lambda_3}{\lambda_4}\right)$ of the group

$$Z\left(\frac{\lambda_2}{\lambda_1}\right) Z\left(\frac{\lambda_3}{\lambda_4}\right) = Z\left(\frac{\lambda_5}{\lambda_6}\right) \tag{6.34}$$

where $Z\left(\frac{\lambda_5}{\lambda_6}\right)$ is another element of the group with $\lambda_5 = \lambda_2\, \lambda_3$ and $\lambda_6 = \lambda_1\, \lambda_4$.

6.5 Summary and Concluding Remarks

Let us now summarize what we have seen. After a very brief look at scale symmetry in classical and quantum mechanics, we reviewed the idea of scale invariance in classical and quantum field theory. The concepts of scale dimension and the conservation of scale current were discussed and we learned why scale symmetry is an *unwanted* symmetry quite unlike other beloved symmetries like Lorentz symmetry and gauge symmetry.

Next we derived the Ward identity of broken scale invariance following standard manipulations utilizing the divergence of scale current. By some examples we learned why it is too naive. We, then, proceeded to learn how to *fix* it. We saw that a divergent wave function renormalization constant leaves its mark in the renormalized theory in the form of *finite* anomalous dimension which signals a deviation from canonical scale dimension. If this was all there is to it, we would again recover scaling behaviour with naive scale dimensions modified by anomalous dimensions.

Following Wilson, we arrived at another, more significant departure from canonical behaviour arising from scale dependence of couplings. This scale dependence is governed by the so-called β function. The last and final modification to the naive Ward identity arose from the need to accommodate this effect. It turns out that β function controls the fate of quantum field theories as one tries to remove the cut-off.

Incorporation of all the changes converts the naive Ward identity to the famous Callan-Symanzik renormalization group equation which uses on-shell subtraction scheme. We saw some drawbacks of on-shell renormalization scheme and saw the usefulness of off-shell subtraction the utility of which was first brought to light by Gell-Mann and Low. We looked at the Gell-Mann Low renormalization equation. For non-abelian gauge theories, perturbative calculations are most conveniently carried out by dimensional regularization and the minimal subtraction scheme. We also took a look at the resulting renormalization group equation first written down (independently) by 't Hooft and Weinberg.

At the very end we tried to understand the *group* in renormalization group. In the particular example we looked at, namely, one loop vacuum polarization with off-shell subtraction and zero electron mass, we do find a group associated with the changes of scale chosen for subtraction. Physics remains invariant under such changes of scale.

We have left out many topics, especially the applications of perturbative renormalization group. We have also left out the most important subject, Wilson formulation of non-perturbative renormalization group. That, however, is the subject of an entirely different lecture series.

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