CHAPTER 5

An Assortment of Effective Hamiltonians

5.1 Introduction

The purpose of this section is to summarize various effective Hamiltonians encountered in various contexts.

5.2 Bloch-Horowitz Perturbation Theory for Effective Hamiltonian

Consider a Hamiltonian H defined at a cutoff Λ. Let us try to lower the cutoff to λ. Let us denote by Q the operator that projects on to all of the states removed when the cutoff is lowered. Let $P = I - Q$. We have

$$Q^2 = Q, \quad P^2 = P, \quad PQ = QP = 0. \tag{5.1}$$

Our purpose is to find an effective Hamiltonian H_{eff} that produces the same eigenvalues in the sub space P as the original Hamiltonian H.

We start from the Schroedinger equation

$$H \mid \psi\rangle = E \mid \psi\rangle. \tag{5.2}$$

Make projections

$$\begin{aligned} PH(P+Q) \mid \psi\rangle &= EP \mid \psi\rangle, & (5.3)\\ QH(P+Q) \mid \psi\rangle &= EQ \mid \psi\rangle. & (5.4) \end{aligned}$$

From Eq. (5.4) we have

$$Q \mid \psi\rangle = \frac{1}{E - QHQ} QHP \mid \psi\rangle. \tag{5.5}$$

Substituting back in Eq. (5.3), we have,

$$H_{eff} P \mid \psi\rangle = EP \mid \psi\rangle, \tag{5.6}$$

where we have defined the effective Hamiltonian

$$H_{eff}(E) = PHP + PHQ \frac{1}{E - QHQ} QHP. \tag{5.7}$$

As this equation shows, H_{eff} is an effective Hamiltonian that produces the same eigenvalues in the subspace P as the original Hamiltonian in the full space. The expression for H_{eff} is not very useful for practical purposes, though. We have simply rewritten the original expression in a more complicated way. Furthermore H_{eff} depends on the unknown energy E.

Towards a perturbative evaluation, divide

$$H = H_0 + V, \tag{5.8}$$

with $[Q, H_0] = 0$. Then

$$H_{eff}(E) = PHP + PVQ\frac{1}{E - QH_0Q - QVQ}QVP. \tag{5.9}$$

Using

$$\frac{1}{A-B} = \frac{1}{A}\sum_{n=0}^{\infty}[B\frac{1}{A}]^n \tag{5.10}$$

we arrive at

$$\begin{aligned} H_{eff}(E) \quad &= PHP + PVQ\frac{1}{E-QH_0Q}QVP + \\ &PVQ\frac{1}{E-QH_0Q}QVQ\frac{1}{E-QH_0Q}QVP + \ldots . \end{aligned} \tag{5.11}$$

Denote the eigen states of H_0 in the subspace of Q by $\mid i\rangle$ with eigen values ϵ_i. Then

$$\begin{aligned} PVQ\frac{1}{E-QH_0Q}QVP \quad &= \frac{1}{2}\sum_i \frac{1}{E-\epsilon_i}PV\mid i\rangle\langle i\mid VP \\ &+ \frac{1}{2}\sum_i PV\mid i\rangle\langle i\mid VP\frac{1}{E-\epsilon_i} \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} PVQ\frac{1}{E-QH_0Q}QVQ\frac{1}{E-QH_0Q}QVP &= \frac{1}{2}\sum_{ij}\frac{1}{E-\epsilon_i}\frac{1}{E-\epsilon_j}PV\mid i\rangle\langle i\mid V\mid j\rangle\langle j\mid VP \\ &+\frac{1}{2}\sum_{ij}PV\mid i\rangle\langle i\mid V\mid j\rangle\langle j\mid VP\frac{1}{E-\epsilon_i}\frac{1}{E-\epsilon_j}. \end{aligned} \tag{5.13}$$

Expanding up to third order in V, we have,

$$\begin{aligned} H_{eff}(E) \quad &= PHP + \frac{1}{2}\sum_i\frac{1}{E-\epsilon_i}PV\mid i\rangle\langle i\mid VP + \frac{1}{2}\sum_i PV\mid i\rangle\langle i\mid VP\frac{1}{E-\epsilon_i} \\ &+\frac{1}{2}\sum_{ij}\frac{1}{E-\epsilon_i}\frac{1}{E-\epsilon_j}PV\mid i\rangle\langle i\mid V\mid j\rangle\langle j\mid VP \\ &+\frac{1}{2}\sum_{ij}PV\mid i\rangle\langle i\mid V\mid j\rangle\langle j\mid VP\frac{1}{E-\epsilon_i}\frac{1}{E-\epsilon_j}. \end{aligned} \tag{5.14}$$

Consider

$$\begin{aligned}\frac{1}{E-\epsilon_i}PV \mid i\rangle\langle i \mid VP &= P\frac{1}{E-\epsilon_i}PV \mid i\rangle\langle i \mid VP \\ &= \frac{1}{PHP-\epsilon_i}PV \mid i\rangle\langle i \mid VP \qquad (5.15)\end{aligned}$$

where in the last line we have replaced the energy eigen value E by the Hamiltonian H itself. Thus we can rewrite

$$\begin{aligned}H_{eff}(E) &= PHP + \frac{1}{2}\sum_i \frac{1}{PHP-\epsilon_i}PV \mid i\rangle\langle i \mid VP + \frac{1}{2}\sum_i PV \mid i\rangle\langle i \mid VP\frac{1}{PHP-\epsilon_i} \\ &+\frac{1}{2}\sum_{ij}\frac{1}{PHP-\epsilon_i}\frac{1}{PHP-\epsilon_j}PV \mid i\rangle\langle i \mid V \mid j\rangle\langle j \mid VP \\ &+\frac{1}{2}\sum_{ij} PV \mid i\rangle\langle i \mid V \mid j\rangle\langle j \mid VP\frac{1}{PHP-\epsilon_i}\frac{1}{PHP-\epsilon_j}. \qquad (5.16)\end{aligned}$$

This is not yet a consistent perturbation theory formula yet, since the right hand side involves PHP itself in the denominators. Hence this is not yet a consistent perturbative expansion up to third order. Using,

$$\begin{aligned}\frac{1}{PHP-\epsilon_i} &= \frac{1}{PH_0P-\epsilon_i+PVP} \\ &= \frac{1}{PH_0P-\epsilon_i} - \frac{1}{PH_0P-\epsilon_i}PVP\frac{1}{PH_0P-\epsilon_i}, \qquad (5.17)\end{aligned}$$

we arrive at the final form

$$\begin{aligned}H^{BH} &= PH_0P + PVP \\ &+\frac{1}{2}\sum_i \frac{1}{PH_0P-\epsilon_i}PV \mid i\rangle\langle i \mid VP \\ &+\frac{1}{2}\sum_i PV \mid i\rangle\langle i \mid VP\frac{1}{PH_0P-\epsilon_i} \\ &-\frac{1}{2}\sum_i \frac{1}{PH_0P-\epsilon_i}PVP\frac{1}{PH_0P-\epsilon_i}PV \mid i\rangle\langle i \mid VP \\ &-\frac{1}{2}\sum_i PV \mid i\rangle\langle i \mid VP\frac{1}{PH_0P-\epsilon_i}PVP\frac{1}{PH_0P-\epsilon_i} \\ &+\frac{1}{2}\sum_{ij}\frac{1}{PH_0P-\epsilon_i}\frac{1}{PH_0P-\epsilon_j}PV \mid i\rangle\langle i \mid V \mid j\rangle\langle j \mid VP \\ &+\frac{1}{2}\sum_{ij} PV \mid i\rangle\langle i \mid V \mid j\rangle\langle j \mid VP\frac{1}{PH_0P-\epsilon_i}\frac{1}{PH_0P-\epsilon_j}. \qquad (5.18)\end{aligned}$$

5.3 Bloch Perturbation Theory for Effective Hamiltonian

Bloch perturbation theory was introduced in Ref. [1] and later reinvented by Wilson[2] in the context of renormalization group. Here we follow the treatment in Ref. [3] where the reader can find many examples of perturbative calculations.

Consider a Hamiltonian H defined at a cutoff Λ. Let us try to lower the cutoff to λ. In general, the cutoff could be in energy and/or particle number. Let us denote by Q the operator that projects on to all of the states removed when the cutoff is lowered. Let $P = I - Q$. We have

$$Q^2 = Q, \quad P^2 = P, \quad PQ = QP = 0. \tag{5.19}$$

Our purpose is to find an effective Hamiltonian H_{eff} that produces the same eigenvalues in the sub space P as the original Hamiltonian H.

Introduce an operator R that satisfies

$$Q \mid \psi\rangle = RP \mid \psi\rangle \tag{5.20}$$

for all eigenstates of the Hamiltonian that have support in the subspace P. R gives the part of $\mid \psi\rangle$ outside the space projected by P in terms of the part of $\mid \psi\rangle$ inside the space. Require that R gives zero acting on states outside the subspace. This means $R = RP$, $R = QR$, $R^2 = 0$. From $R = QR$, we have, $PR = 0$. Note also that $R^\dagger \neq R$.

Start from the set of equations

$$PHP \mid \psi\rangle + PHQ \mid \psi\rangle \quad = EP \mid \psi\rangle, \tag{5.21}$$

$$QHP \mid \psi\rangle + QHQ \mid \psi\rangle \quad = EQ \mid \psi\rangle. \tag{5.22}$$

From Eq. (5.21),

$$RPHP \mid \psi\rangle + RPHQRP \mid \psi\rangle = ERP \mid \psi\rangle. \tag{5.23}$$

From Eq. (5.22),

$$QHP \mid \psi\rangle + QHQRP \mid \psi\rangle = ERP \mid \psi\rangle. \tag{5.24}$$

Subtracting,

$$RH_{PP} - H_{QQ}R + RH_{PQ}R - H_{QP} = 0. \tag{5.25}$$

We have introduced the notations, $PHP = H_{PP}$ and so on. Put $H = h + v$ with $[h, Q] = 0$. Then

$$Rh_{PP} - h_{QQ}R - v_{QP} + Rv_{PP} - v_{QQ}R + Rv_{PQ}R = 0 \tag{5.26}$$

which shows that R starts first order in v.

We start from the eigenvalue equation,

$$H(P + Q) \mid \psi\rangle = E(P + Q) \mid \psi\rangle. \tag{5.27}$$

i.e.,

$$H(P + R)P \mid \psi\rangle = E(P + R)P \mid \psi\rangle. \tag{5.28}$$

Multiplying from the left by $(P+R^\dagger)$ we have,

$$(P+R^\dagger)H(P+R)P\mid\psi\rangle = E(P+R^\dagger)(P+R)P\mid\psi\rangle. \tag{5.29}$$

Using $PR=0$, $R^\dagger P=0$, $(P+R^\dagger)(P+R)=P+R^\dagger R$. Thus we can rewrite the eigenvalue equation as

$$\Big[\frac{1}{1+R^\dagger R}\Big]^{\frac{1}{2}}(P+R^\dagger)H(P+R)\Big[\frac{1}{1+R^\dagger R}\Big]^{\frac{1}{2}}[1+R^\dagger R]^{\frac{1}{2}}P\mid\psi\rangle = E[1+R^\dagger R]^{\frac{1}{2}}P\mid\psi\rangle. \tag{5.30}$$

i.e.,

$$H_{eff}\mid\phi\rangle = E\mid\phi\rangle \tag{5.31}$$

where

$$\mid\phi\rangle = [1+R^\dagger R]^{\frac{1}{2}}P\mid\psi\rangle \tag{5.32}$$

and

$$H_{eff} = \Big[\frac{1}{1+R^\dagger R}\Big]^{\frac{1}{2}}(P+R^\dagger)H(P+R)\Big[\frac{1}{1+R^\dagger R}\Big]^{\frac{1}{2}}. \tag{5.33}$$

Our next task is to generate a perturbative expansion. Denote free eigenstates in P by $\mid a\rangle$, $\mid b\rangle$, etc. Denote free eigenstates in Q by $\mid i\rangle$, $\mid j\rangle$, etc. Then

$$\begin{aligned} h_{PP}\mid a\rangle &= \epsilon_a\mid a\rangle,\\ h_{QQ}\mid i\rangle &= \epsilon_i\mid i\rangle. \end{aligned} \tag{5.34}$$

Let us compute R to lowest orders in the perturbation theory. Let us write $R=R_1+R_2+\ldots$ where the subscript denotes orders in v. A straightforward calculation leads to

$$\langle i\mid R_1\mid a\rangle = \frac{\langle i\mid v_{QP}\mid a\rangle}{\epsilon_a-\epsilon_i}, \tag{5.35}$$

$$\langle i\mid R_2\mid a\rangle = -\sum_b\frac{\langle b\mid v\mid a\rangle\langle i\mid v\mid b\rangle}{(\epsilon_a-\epsilon_i)(\epsilon_b-\epsilon_i)} + \sum_j\frac{\langle i\mid v\mid j\rangle\langle j\mid v\mid a\rangle}{(\epsilon_a-\epsilon_i)(\epsilon_a-\epsilon_j)}. \tag{5.36}$$

Our next task is to develop a perturbation theory expansion for the effective Hamiltonian to a given order.

We start from the expression for the effective Hamiltonian. Remember that $R_1\sim O(v)$, $R_2\sim O(v^2)$.

To order v, $H_{eff}=PHP$ and hence

$$\langle a\mid H_{eff}\mid b\rangle = \langle a\mid (h+v)\mid b\rangle. \tag{5.37}$$

To second order in v, we have

$$H_{eff} = [1-\frac{1}{2}R^\dagger R][PHP+PHR+R^\dagger HP+R^\dagger HR][1-\frac{1}{2}R^\dagger R]. \tag{5.38}$$

From $R^\dagger HR$ we get,

$$\langle a \mid R^\dagger HR \mid b\rangle = \sum_i \epsilon_i \frac{\langle a \mid v \mid i\rangle\langle i \mid v \mid b\rangle}{(\epsilon_a - \epsilon_i)(\epsilon_b - \epsilon_i)}. \tag{5.39}$$

From PHR and $R^\dagger HP$ terms we get

$$\sum_i \langle a \mid H \mid i\rangle\langle i \mid R_1 \mid b\rangle + \sum_i \langle a \mid R_1^\dagger \mid i\rangle\langle i \mid H \mid b\rangle \tag{5.40}$$

$$= \sum_i \Big[\frac{\langle a \mid v \mid i\rangle\langle i \mid v \mid b\rangle}{\epsilon_a - \epsilon_i} + \frac{\langle a \mid v \mid i\rangle\langle i \mid v \mid b\rangle}{\epsilon_b - \epsilon_i}. \tag{5.41}$$

From the *normalization factors* we get

$$-\frac{1}{2}R^\dagger RPHP - \frac{1}{2}PHPR^\dagger R = -\frac{1}{2}(\epsilon_a + \epsilon_b)\sum_i \frac{\langle a \mid v \mid i\rangle\langle i \mid v \mid b\rangle}{(\epsilon_a - \epsilon_i)(\epsilon_b - \epsilon_i)} \tag{5.42}$$

Adding everything, to second order, we have,

$$\langle a \mid H_{eff} \mid b\rangle = \frac{1}{2}\sum_i \langle a \mid v \mid i\rangle\langle i \mid v \mid b\rangle \Big[\frac{1}{\epsilon_a - \epsilon_i} + \frac{1}{\epsilon_b - \epsilon_i}\Big]. \tag{5.43}$$

If $a = b$, this expression reduces to the familiar second order energy shift.

Why Bloch formalism is preferred over Bloch-Horowitz formalism?

In the former, eigenstates of the effective Hamiltonian are ortho normalized projections of the original eigenstates. In the latter, they are not.

Consider two ortho normalized eigenstates of the original Hamiltonian $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ with $\langle \psi_1 \mid \psi_2\rangle = 0$. However, $P \mid \psi_1\rangle$ and $P \mid \psi_2\rangle$ need not be orthogonal, i.e., $\langle \psi_1 \mid PP \mid \psi_2\rangle = \langle \psi_1 \mid P \mid \psi_2\rangle \neq 0$. Consider

$$\begin{aligned}\langle \psi_1 \mid \psi_2\rangle \quad &= \langle \psi_1 \mid P \mid \psi_2\rangle + \langle \psi_1 \mid Q^2 \mid \psi_2\rangle \\ &= \langle \psi_1 \mid P \mid \psi_2\rangle + \langle \psi_1 \mid P^\dagger R^\dagger RP \mid \psi_1\rangle. \end{aligned} \tag{5.44}$$

Construct $\mid \tilde{\psi}_1\rangle = [1 + R^\dagger R]^{\frac{1}{2}} P \mid \psi_1\rangle$, $\mid \tilde{\psi}_2\rangle = [1 + R^\dagger R]^{\frac{1}{2}} \mid \psi_2\rangle$. Then

$$\langle \tilde{\psi}_1 \mid \tilde{\psi}_2\rangle = \langle \psi_1 \mid P \mid \psi_2\rangle + \langle \psi_1 \mid PR^\dagger RP \mid \psi_2\rangle = \langle \psi_1 \mid \psi_2\rangle. \tag{5.45}$$

5.4 Similarity Perturbation Theory for Effective Hamiltonian

Realizing that a single step diagonalization of the Hamiltonian is too difficult, one may aim for the next best thing, namely, bringing the Hamiltonian to a band diagonal form. This is the method proposed by Glazek and Wilson and Wegner independently[4].

Starting from a cutoff Hamiltonian H_B which includes canonical terms and counter terms we wish to arrive at an effective Hamiltonian H_σ defined at the scale σ via a similarity transformation

$$H_\sigma = S_\sigma \; H_B \; S_\sigma^\dagger \tag{5.46}$$

where S_σ is chosen to be unitary.

The boundary condition is $Limit_{\sigma \to \infty}\ H_\sigma = H_B$.

Introduce anti-Hermitian generator of infinitesimal changes of scale T_σ through

$$S_\sigma = \mathcal{T}\ e^{\int_\sigma^\infty d\sigma'\ T_{\sigma'}} \tag{5.47}$$

where $\mathcal{T}$ puts operators in order of increasing scale.

For infinitesimal change of scale, $S_\sigma = 1 - T_\sigma\ d\sigma$ and $S_\sigma^\dagger = 1 + T_\sigma\ d\sigma$. Then we arrive at the infinitesimal form of the transformation

$$\frac{dH_\sigma}{d\sigma} = [H_\sigma, T_\sigma]. \tag{5.48}$$

This equation which has been called the flow equation of the Hamiltonian is the starting point of the investigations.

The basic goal of the transformation S_σ is that H_σ should be band diagonal relative to the scale σ. Qualitatively this means that matrix elements of H_σ involving energy jumps much larger than σ should be zero. T_σ still remains arbitrary to a great extent. It is instructive to go through the steps of the derivation which leads to the Glazek-Wilson choice.

We write $H_B = H_{B0} + H_{BI}$ where H_{B0} is the free part and H_{BI} is the interaction part of the bare cutoff Hamiltonian. A brute force way of achieving our goal is to *define* the matrix elements $H_{I\sigma ij} = f_{\sigma ij} H_{BIij}$ where we have introduced the function $f_{\sigma ij} = f(x_{\sigma ij})$ with x a function of σ^2 and ΔM_{ij}^2. The function $f(x)$ should be chosen as follows:

$$\begin{array}{llll} \text{when } \sigma^2 >> \Delta M_{ij}^2, & f(x) = 1 & & (\textit{near diagonal region}); \\ \text{when } \sigma^2 << \Delta M_{ij}^2, & f(x) = 0 & & (\textit{far off diagonal region}); \\ \text{in between} & f(x) & \text{drops from 1 to 0} & (\textit{transition region}). \end{array} \tag{5.49}$$

Here $\Delta M_{ij}^2 (= M_i^2 - M_j^2)$ denotes the difference of invariant masses of states i and j. Because of the properties of f, $H_{I\sigma ij}$ is band diagonal. What is wrong with such a choice of inserting form factors by hand at the interaction vertices? First of all, we have simply discarded degrees of freedom above σ. Secondly, H_σ will have very strong dependence on σ. Thirdly, to ensure that H_σ has no ultraviolet cutoff dependence, H_B should contain canonical and counterterms. But, in light front Hamiltonian field theory, because of the complexities due to renormalization, a priori we do not know the structure of counterterms.

Note that in the definition of H_σ given in Eq. (5.48) the form of T_σ is still unspecified. In fact, a wide variety of choices are possible. In the following, we consider the choices made by Głazek and Wilson and Wegner. The price we have to pay for the use of flow equations is that it will generate complicated interactions even if the starting Hamiltonian has only simple interactions. For example, starting with a Hamiltonian which has only 2 particle interaction, the transformation will generate 3 particle interactions, 4 particle interactions, etc.

5.4.1 Glazek-Wilson Formalism

In this section we follow the treatment in Ref. [5]

Writing $H_\sigma = H_0 + H_{I\sigma}$, noting that the free Hamiltonian H_0 does not depend on σ and taking matrix elements in free particle states, we have,

$$[H_\sigma, T_\sigma]_{ij} = (P_i^- - P_j^-)T_{\sigma ij} + [H_{I\sigma}, T_\sigma]_{ij} \tag{5.50}$$

where $H_0 \mid i\rangle = P_i^- \mid i\rangle$, etc. i.e.,

$$\frac{1}{f_{\sigma ij}}\frac{dH_{I\sigma ij}}{d\sigma} = \frac{1}{f_{\sigma ij}}[H_{I\sigma}, T_\sigma]_{ij} + \frac{1}{f_{\sigma ij}}(P_i^- - P_j^-)T_{\sigma ij}. \tag{5.51}$$

Since we want $H_{I\sigma ij}$ to be band diagonal, it is advantageous to trade the factor $\frac{1}{f_{\sigma ij}}\frac{dH_{I\sigma ij}}{d\sigma}$ for $\frac{d}{d\sigma}\left[\frac{1}{f_{\sigma ij}}H_{I\sigma ij}\right]$ which on integration has the chance to ensure that $H_{I\sigma ij}$ is band diagonal, we use

$$\frac{d}{d\sigma}\left[\frac{1}{f_{\sigma ij}}H_{I\sigma ij}\right] + \frac{1}{f_{\sigma ij}^2}\frac{df_{\sigma ij}}{d\sigma}H_{I\sigma ij} = \frac{1}{f_{\sigma ij}}\frac{dH_{I\sigma ij}}{d\sigma} \tag{5.52}$$

and arrive at

$$\begin{aligned}\frac{d}{d\sigma}\left[\frac{1}{f_{\sigma ij}}H_{I\sigma ij}\right] &= \frac{1}{f_{\sigma ij}}(P_i^- - P_j^-)T_{\sigma\ ij} \\ &+ \frac{1}{f_{\sigma ij}}[H_{I\sigma}, T_\sigma]_{ij} - \frac{1}{f_{\sigma ij}^2}\frac{df_{\sigma ij}}{d\sigma}H_{I\sigma ij}.\end{aligned} \tag{5.53}$$

Still $T_{\sigma ij}$ is not defined. We next convert this equation into two equations, one define defining the flow of $H_{I\sigma ij}$ and other defining $T_{\sigma ij}$. Recalling the starting equation Eq. (5.48) we add and subtract $[H_{I\sigma}, T_\sigma]_{ij}$ to the r.h.s and arrive at

$$\begin{aligned}\frac{d}{d\sigma}\left[\frac{1}{f_{\sigma ij}}H_{I\sigma ij}\right] &= [H_{I\sigma}, T_\sigma]_{ij} + \frac{1}{f_{\sigma ij}}(P_i^- - P_j^-)T_{\sigma\ ij} \\ &+ \frac{1}{f_{\sigma ij}}(1 - f_{\sigma ij})[H_{I\sigma}, T_\sigma]_{ij} - \frac{1}{f_{\sigma ij}^2}\frac{df_{\sigma ij}}{d\sigma}H_{I\sigma ij}.\end{aligned} \tag{5.54}$$

Głazek and Wilson choose T_σ to be

$$T_{\sigma ij} = \frac{1}{P_j^- - P_i^-}\left[(1 - f_{\sigma ij})[H_{I\sigma}, T_\sigma]_{ij} - \frac{d}{d\sigma}(\ln\, f_{\sigma ij})H_{I\sigma ij}\right]. \tag{5.55}$$

Then from Eq. (5.54), we have,

$$\frac{dH_{I\sigma ij}}{d\sigma} = f_{\sigma ij}[H_{I\sigma}, T_\sigma]_{ij}. \tag{5.56}$$

Integrating Eq. (5.56) from σ to ∞, we arrive at,

$$H_{I\sigma ij} = f_{\sigma ij}\left[H_{IBij} - \int_\sigma^\infty d\sigma'[H_{I\sigma'}, T_{\sigma'}]_{ij}\right]. \tag{5.57}$$

Note that $H_{I\sigma ij}$ is zero in the far-off diagonal region. This is clear from the solution given in Eq. (5.57) since $f(x)$ vanishes when $x \geq 2/3$.

$T_{\sigma ij}$ vanishes in the near diagonal region. When i is close to j, $f_{\sigma ij} = 1$ and both $(1 - f_{\sigma ij})$ and $\frac{d}{d\sigma} f_{\sigma ij}$ vanishes. It follows, then, from Eq. (5.55) that $T_{\sigma ij}$ vanishes in the near-diagonal region. This guarantees that a perturbative solution to $H_{I\sigma ij}$ in terms of H_{BIij} will never involve vanishing energy denominators.

Our next task is to derive the effective Hamiltonian to second order in perturbation theory. Using

$$H_{I\sigma ik} \simeq f_{\sigma ik} H_{BIik} \tag{5.58}$$

and

$$T_{\sigma kj} \simeq \frac{1}{P_j^- - P_k^-} \left\{ -\frac{d}{d\sigma} (\ln\, f_{\sigma kj}) f_{\sigma kj} H_{BIkj} \right\} \tag{5.59}$$

in Eq. (5.57), a straightforward calculation leads to

$$H^{(2)}_{I\sigma ij} = - \sum_k H_{BIik} H_{BIkj} \left[\frac{g_{\sigma ijk}}{P_k^- - P_j^-} + \frac{g_{\sigma jik}}{P_k^- - P_i^-} \right], \tag{5.60}$$

where

$$\begin{aligned} g_{\sigma ijk} &= f_{\sigma ij} \int_\sigma^\infty d\sigma' \; f_{\sigma' ik} \frac{d}{d\sigma'} f_{\sigma' \; jk}, \\ g_{\sigma jik} &= f_{\sigma ij} \int_\sigma^\infty d\sigma' \; f_{\sigma' jk} \frac{d}{d\sigma'} f_{\sigma' ik}. \end{aligned} \tag{5.61}$$

We find that the effective Hamiltonian in similarity perturbation theory is a modification of the effective Hamiltonian in Bloch perturbation theory[3].

5.4.2 Wegner Formalism

In the Wegner formalism[6], the flow equation is given by

$$\frac{dH(l)}{dl} = [\tau(l), H(l)]. \tag{5.62}$$

Wegner chooses

$$\tau(l) = [H_d, H] = [H_d, H_r] \tag{5.63}$$

where H_d is the diagonal part of the Hamiltonian and H_r is the rest, i.e., $H = H_d + H_r$. Here the word diagonal is used in the particle number conserving sense. It is important to note that H_d is not the free part of the Hamiltonian and both H_d and H_r depend on the length scale l.

The light front Hamiltonian has dimension of $(mass)^2$ and hence τ has the dimension of $(mass)^4$, l has dimension of $\frac{1}{(mass)^4}$.

Expanding in powers of the coupling constant,

$$H = H_d^{(0)} + H_r^{(1)} + H_d^{(2)} + H_r^{(2)} + \ldots \tag{5.64}$$

where the superscript denotes the order in the coupling constant,

$$\tau(l) = [H_d^{(0)}, H_r^{(1)}] + [H_d^{(0)}, H_r^{(2)}] + \ldots \ . \tag{5.65}$$

Then, to second order,

$$\frac{dH}{dl} = [[H_d^{(0)}, H_r^{(1)}], H_d^{(0)}] + [[H_d^{(0)}, H_r^{(1)}], H_r^{(1)}] + [[H_d^{(0)}, H_r^{(2)}], H_d^{(0)}] + \ldots \ . \tag{5.66}$$

Introduce[7] the eigenstates of $H_d^{(0}$,

$$H_d^{(0)} \mid i\rangle = P_i^- \mid i\rangle. \tag{5.67}$$

Then, to second order,

$$\frac{dH_{lij}}{dl} = -(P_i^- - P_j^-)^2 H_{rij}^{(1)} + [\tau_l^{(1)}, H_r^{(1)}]_{ij} - (P_i^- - P_j^-)^2 H_{rij}^{(2)} + \ldots \ . \tag{5.68}$$

To first order in the coupling,

$$\frac{dH_{rij}}{dl} = -(P_i^- - P_j^-)^2 H_{rij}^{(1)} \tag{5.69}$$

which on integration yields

$$H_{rij}^{(1)}(\sigma) = e^{-\frac{(P_i^- - P_j^-)^2}{\sigma^4}} H_{rij}^1(\Lambda) \tag{5.70}$$

where we have introduced the energy scale $l = \frac{1}{\sigma^4}$ and used the fact that $l = 0$ corresponds to the original bare cutoff. We notice the emergence of the similarity factor $f_{\sigma ij} = e^{-\frac{(P_i^- - P_j^-)^2}{\sigma^4}}$.

If we are interested only in particle number conserving (diagonal) part of the effective interaction, we have to second order,

$$\frac{dH_{lij}}{dl} = [\tau_l^{(1)}, H_r^{(1)}]_{ij} \tag{5.71}$$

Using

$$\tau_{lij}^{(1)} = (P_j^- - P_i^-) H_{rij}^{(1)}, \tag{5.72}$$

the second order generated effective interaction in the diagonal sector is

$$H_{lij} = \sum_k H_{ik}^B H_{kj}^B \frac{(P_i^- - P_k^-) + (P_j^- - P_k^-)}{(P_i^- - P_k^-)^2 + (P_j^- - P_k^-)^2} \left[1 - e^{-\left\{(P_i^- - P_k^-)^2 + (P_j^- - P_k^-)^2\right\}/\sigma^4}\right]. \tag{5.73}$$

Even though the second order formula is very similar to the one in Glazek-Wilson formalism when an exponential form is chosen for the similarity factor (see Sec. IV), we note a slight difference. In the Glazek-Wilson formalism, since the purpose is to bring the Hamiltonian into a band diagonal form, even in the particle number conserving sectors the large jumps in energies do not appear by construction. In the version of the Wegner formalism presented here the purpose is to bring the Hamiltonian in the block diagonal form in particle number sector so that large jumps in energies are allowed by the effective Hamiltonian. Note that small energy denominators do not appear in both formalisms.

BIBLIOGRAPHY

[1] C. Bloch, Nucl. Phys. **6**, 329 (1958).

[2] K. G. Wilson, Phys. Rev. D **2**, 1438 (1970).

[3] R. J. Perry, Annals Phys. **232** (1994) 116 [hep-th/9402015]; B. D. Jones and R. J. Perry, Phys. Rev. D **55**, 7715 (1997).

[4] S. D. Glazek and K. G. Wilson, Phys. Rev. D **48**, 5863 (1993); **49**, 4214 (1994); Similar flow equations were proposed by F. Wegner, Ann. Phys. (Leipzig) **3**, 77 (1994).

[5] K. G. Wilson, T. S. Walhout, A. Harindranath, W. Zhang, R. J. Perry and S. D. Glazek, Phys. Rev. D **49**, 6720 (1994) [hep-th/9401153].

[6] For a summary of recent work in condensed matter physics and quantum field theory using the Wegner formalism see F. J. Wegner, Nucl. Phys. Proc. Suppl. **90**, 141 (2000).

[7] E. L. Gubankova and F. Wegner, odified similarity renormalization of Hamiltonians: Positronium on the hep-th/9708054; E. L. Gubankova and F. Wegner, Phys. Rev. D **58**, 025012 (1998) [hep-th/9710233].