CHAPTER 4

Discretized Light Cone Quantization (DLCQ): An Introduction

In this chapter we discuss basic features of discretized light cone quantization (DLCQ) approach to light front field theory[1, 2, 3, 4] with specific examples. For simplicity we restrict ourselves to 1+1 dimensions and discuss features of the 3+1 dimensional case towards the very end of this section. We follow the treatment in Ref. [5].

As an illustrative example we consider two-dimensional ϕ^4 theory. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{4.1}$$

The equation of motion is

$$\partial^+\partial^-\phi + m^2\phi + \frac{\lambda}{3!}\phi^3 = 0. \tag{4.2}$$

Let us put the system in a box $-L \leq x^- \leq +L$. We can choose to impose either periodic or anti-periodic boundary conditions. As we shall see later on, the structure of the discretized theory looks drastically different with these different boundary conditions. Let us first consider antiperiodic boundary condition.

4.1 Anti-Periodic Boundary Condition

4.1.1 Quantization

We perform a mode expansion

$$\phi(x) = \frac{1}{(2L)^{\frac{1}{2}}}\sum_n q_n(x^+)e^{-\frac{i}{2}k_n^+x^-} \tag{4.3}$$

with $k_n^+ = \frac{n\pi}{L}$, $n = \pm1, \pm3, \pm5, \dots$. The Lagrangian L is

$$L = \sum_n \frac{1}{2}\dot{q}_n q_{-n} i k_n^+ - \frac{1}{2}m^2\sum_n q_n q_{-n} - \frac{\lambda}{4!}\frac{1}{2L}\sum_{nmpq} q_n q_m q_p q_q \delta_{n+m+p+q,0}, \tag{4.4}$$

with $\dot{q}_n = \partial^- q_n$. The conjugate momenta $p_n = \frac{\partial L}{\partial \dot{q}_n} = \frac{i}{2}k_n^+ q_{-n}$. We see that p_n and q_n are not independent. Following Dirac[6], introduce the primary constraint

$$C_n = p_n - \frac{i}{2}k_n^+ q_{-n} \approx 0. \tag{4.5}$$

The canonical Hamiltonian

$$P_c^- = \sum_n p_n \dot{q}_n - L = \frac{1}{2} m^2 \sum_n q_n q_{-n} + \frac{\lambda}{4!} \sum_{nmpq} q_n q_m q_p q_q \delta_{n+m+p+q,0}. \tag{4.6}$$

The primary Hamiltonian

$$P_p^- = P_c^- + \sum_n v_n C_n \tag{4.7}$$

where v_n are the Lagrange multipliers.

Let us define the Poisson Bracket for any two variables A and B,

$$[A, B]_{PB} = \sum_m \Big[\frac{\partial A}{\partial q_m} \frac{\partial B}{\partial p_m} - \frac{\partial A}{\partial p_m} \frac{\partial B}{\partial q_m} \Big]. \tag{4.8}$$

Thus we have, $[q_i, q_j]_{PB} = 0, [p_i, p_j]_{PB} = 0, [q_i, p_j]_{PB} = \delta_{ij}$. For the Poisson bracket of the primary constraints we have,

$$[C_m, C_n]_{PB} = i k_n^+ \delta_{n+m,0}. \tag{4.9}$$

We have

$$\dot{C}_n = [C_n, P_p^-]_{PB} = i k_n^+ v_{-n} - \Big[m^2 q_{-n} + \frac{\lambda}{3!} \frac{1}{2L} \sum_{qrs} q_q q_r q_s \delta_{n+q+r+s,0} \Big]. \tag{4.10}$$

From $\dot{C}_n = 0$, we get,

$$i k_n^+ v_{-n} = -m^2 q_{-n} - \frac{\lambda}{3!} \frac{1}{2L} \sum_{qrs} q_q q_r q_s \delta_{n+q+r+s,0} \tag{4.11}$$

which determines the Lagrange multiplier v_n for all n. Thus there are no secondary constraints. The constraints we have are second class. The constraint matrix has elements $\mathcal{C}_{mn} = i k_n^+ \delta_{m+n,0}$ and the inverse is given by $\mathcal{C}^{-1}_{mn} = \frac{1}{i k_m^+} \delta_{m+n,0}$ We have $[q_m, q_n]_{PB} = 0, [q_m, C_p]_{PB} = \delta_{m,p}$. Thus

$$\begin{aligned} [q_m, q_n]_{DB} &= [q_m, q_n]_{PB} - [q_m, C_p]_{PB} \mathcal{C}^{-1}_{pq} [C_q, q_n]_{PB} \\ &= -\delta_{m,p} \frac{1}{i k_p^+} \delta_{p+q,0} (-) \delta_{q,n} \\ &= \frac{1}{i m \frac{\pi}{L}} \delta_{m+n,0}. \end{aligned} \tag{4.12}$$

The quantum commutator bracket is given by

$$[q_m, q_n] = i [q_m, q_n]_{DB} = \frac{L}{m\pi} \delta_{m+n,0}. \tag{4.13}$$

Defining $q_m = \left[\frac{L}{m\pi}\right]^{\frac{1}{2}} a_m$ and $q_{-m} = \left[\frac{L}{m\pi}\right]^{\frac{1}{2}} a_m^\dagger$ for $m > 0$, we have

$$\phi(x^-) = \frac{1}{\sqrt{2\pi}} \sum_{m=1,3,5,\ldots} \frac{1}{\sqrt{m}} \Big[a_m e^{-\frac{i}{2} \frac{\pi}{L} m x^-} + a_m^\dagger e^{\frac{i}{2} \frac{\pi}{L} m x^-} \Big]. \tag{4.14}$$

with $[a_m, a_n] = 0$, $[a_m^\dagger, a_n^\dagger] = 0$, $[a_n, a_m^\dagger] = \delta_{n,m}$.

An explicit calculation leads to the equal light-front time commutator of fields

$$[\phi(x^+, x^-), \phi(x^+, y^-)] = -\frac{i}{4}\epsilon(x^- - y^-), \tag{4.15}$$

where the antisymmetric step function ϵ is given by

$$\epsilon(x^-) = \frac{2}{i\pi}\sum_{1,3,\ldots}\frac{1}{n}\Big[e^{i\frac{\pi}{L}nx^-} - e^{-i\frac{\pi}{L}nx^-}\Big]. \tag{4.16}$$

4.1.2 Free theory

Let us consider the momentum operator P^+ and the Hamiltonian operator P^- in free scalar theory. We have

$$P^+ = \frac{1}{2}\int_{-L}^{+L} dx^- \partial^+\phi\partial^+\phi \tag{4.17}$$

and

$$P^- = \frac{1}{2}m^2\int_{-L}^{+L} dx^- \phi^2 \tag{4.18}$$

Using the field expansion given in Eq. (4.14) we arrive at

$$P^+ = \frac{\pi}{L}\sum_{n=1,3,5,\ldots} n a_n^\dagger a_n = \frac{\pi}{L}K \tag{4.19}$$

and

$$P^- = \frac{L}{\pi}m^2\sum_{n=1,3,5,\ldots}\frac{1}{n}a_n^\dagger a_n = \frac{L}{\pi}H. \tag{4.20}$$

We have the mass operator $M^2 = P^+P^- = KH$. Let us introduce the Fock space basis with basis states denoted by $\mid n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \ldots\rangle$. This state contains m_1 particles carrying n_1 units of momenta, m_2 particles carrying n_2 units of momenta, and so on. Because of the positivity of longitudinal momentum, we have, $K = m_1n_1 + m_2n_2 + m_3n_3 + \ldots$.

Let us consider a few examples.

K=1

Only one state $\mid 1\rangle \equiv \mid 1^1\rangle$ is possible. We have $\langle 1 \mid M^2 \mid 1\rangle = \langle 1 \mid KH \mid 1\rangle = m^2$. Introducing the boost invariant momentum fraction $x = \frac{n}{K}$, we see that we have a single particle state with momentum fraction 1.

K=2

Only one state $\mid 1\rangle \equiv \mid 1^2\rangle$ is possible. We have two particle with momentum fraction $\frac{1}{2}$ each. We have $\langle 1 \mid M^2 \mid 1\rangle = 4m^2$ as expected.

K=3

Two states are possible. The state $\mid 1\rangle \equiv\mid 3^1\rangle$ is a single particle state with momentum fraction 1. The state $\mid 2\rangle \equiv\mid 1^3\rangle$ is a three particle state with each particle carrying momentum fraction $\frac{1}{3}$. We have $\langle 1 \mid M^2 \mid 1\rangle = m^2$ and $\langle 2 \mid M^2 \mid 2\rangle = 9m^2$.

The resolution K serves to restrict the maximum number of particles allowed. It also determines the minimum momentum fraction allowed for a particle. Since x varies from 0 to 1 continuously in the real world, we find that the continuum limit of DLCQ is arrived by taking the limit $K \to \infty$.

4.2 Periodic Boundary Condition

With periodic boundary condition, zero momentum mode is included in the mode expansion. This introduces nontrivial effects from the very beginning in the discretized theory (the "zero mode problem") as follows.

4.2.1 Zero mode problem

With periodic boundary condition, we can write

$$\phi(x) = \phi_0 + \Phi(x). \tag{4.21}$$

Here ϕ_0 is the zero mode and $\Phi(x)$ contains non-zero modes, i.e., $\int_{-L}^{+L} dx^- \, \Phi(x) = 0$. With periodic boundary condition, $\phi(x^+, -L) = \phi(x^+, +L)$, we have, $\frac{\partial\phi(x^+,-L)}{\partial x^+} = \frac{\partial\phi(x^+,+L)}{\partial x^+}$. Thus we have $\int_{-L}^{+L} dx^- \partial^+\partial^- \phi = 0$. Thus integrating the equation of motion we arrive at

$$m^2 \int_{-L}^{+L} dx^- \phi = -\frac{\lambda}{3!} \int_{-L}^{+L} dx^- \phi^3. \tag{4.22}$$

What is the meaning of this equation? Explicitly, in terms of zero and non-zero modes, the above equation reads

$$\phi_0 = -\frac{1}{2L}\frac{\lambda}{m^2}\frac{1}{3!}\Big[2L\phi_0^3 + \int_{-L}^{+L} dx^- \Big\{\phi_0\Phi^2 + \Phi^2\phi_0 + \Phi\phi_0\Phi + \Phi^3\Big\}\Big]. \tag{4.23}$$

In the free theory, zero mode is zero. In the interacting theory, ϕ_0 is not a dynamical variable (it's time evolution is determined by non-zero modes). In the classical theory the zero mode can be determined in principle by solving an algebraic equation. In the quantum theory, however, one has to solve a non-linear operator equation. Furthermore we need to know the commutation relation of the zero mode with non-zero modes.

What operators get affected by the zero mode? The longitudinal momentum operator involves only partial derivative of ϕ and hence is unaffected by the zero mode. However, the Hamiltonian involves both zero mode and non-zero modes. Thus we have reached the remarkable conclusion that with periodic boundary conditions, the Hamiltonian of the theory is not known unless one has solved for the zero mode!

4.2.2 Quantization

Again as in the case of antiperiodic boundary condition, we start from the Lagrangian density and put the system in box $-L \leq x^- \leq +L$. With periodic boundary condition, the the field expansion takes the form

$$\phi(x) = \frac{1}{\sqrt{2L}} \sum_n q_n(x^+) e^{-\frac{i}{2} k_n^+ x^-} \tag{4.24}$$

with $k_n^+ = \frac{2\pi}{L} n$, $n = 0, \pm 1, \pm 2, \ldots$. The Lagrangian has the *same form* as in Eq. (4.4). The conjugate momenta p_n is determined as before and we end up with a primary constraint given in Eq. (4.5). However this equation in the present case includes the zero mode. We also note that the momentum conjugate to the zero mode p_0 =0. We construct the primary Hamiltonian incorporating the primary constraint by means of Lagrange multipliers v_n and arrive at Eq. (4.11). This equation determines the non-zero modes of v_n but the zero mode of v_n is not determined since $k_0^+ = 0$.

To determine the zero mode of v_n we proceed as follows. Putting $n = 0$, in Eq. (4.11) we get

$$m^2 q_0 + \frac{\lambda}{3!} \frac{1}{2L} \sum_{qrs} q_q q_r q_s \delta_{q+r+s,0} = 0. \tag{4.25}$$

Define a secondary constraint

$$C_0 = m^2 q_0 + \frac{\lambda}{3!} \frac{1}{2L} \sum_{qrs} q_q q_r q_s \delta_{q+r+s,0} \approx 0. \tag{4.26}$$

Now we demand,

$$\dot{C}_0 = [C_0, P_p^-]_{PB} = 0 \tag{4.27}$$

which leads to

$$m^2 v_0 + \frac{\lambda}{2!} \frac{1}{2L} \sum_{irs} v_i q_r q_s \delta_{q+r+i,0} = 0 \tag{4.28}$$

which determines the zero mode of the Lagrange multiplier. Thus we have determined the Lagrange multiplier v_n completely and there are no more constraints. Since C_0 does not contain p_n, we have

$$[C_0, C_0]_{PB} = 0, \quad [q_m, C_0]_{PB} = 0 \tag{4.29}$$

The nonzero Poisson bracket is

$$[C_0, C_n]_{PB} = \frac{\lambda}{2!} \frac{1}{2L} \sum_{rs} q_r q_s \delta_{r+s+n,0} \quad \text{for } n \neq 0. \tag{4.30}$$

Now the constraint matrix depends on the interaction and its inversion is problematic.

4.2.3 Free theory

In the free theory zero mode is unharmful and we can proceed as we did with antiperiodic boundary condition. We arrive at the commutator of non-zero modes

$$[q_m, q_n] = \frac{1}{k_m^+}\delta_{m+n,0} \quad \text{with } k_n^+ = \frac{2\pi}{L}n. \tag{4.31}$$

Defining

$$q_n = \left[\frac{L}{2\pi n}\right]^{\frac{1}{2}} a_n, \quad q_{-n} = \left[\frac{L}{2\pi n}\right]^{\frac{1}{2}} a_n^\dagger, \ n > 0 \tag{4.32}$$

we arrive at

$$\Phi(x) = \frac{1}{\sqrt{4\pi}} \sum_{n=1,2,\ldots} \frac{1}{\sqrt{n}} \left[a_n e^{-i\frac{\pi}{L}nx^-} + a_n^\dagger e^{i\frac{\pi}{L}nx^-}\right] \tag{4.33}$$

with

$$[a_m, a_n^\dagger] = \delta_{mn}, \quad [a_m, a_n] = 0, \quad [a_m^\dagger, a_n^\dagger] = 0. \tag{4.34}$$

Next let us look at the spectrum of free field theory. As before we can construct the operators K and H. We have

$$P^+ = \frac{2\pi}{L}K, \quad \text{with } K = \sum_{1,2,\ldots} n a_n^\dagger a_n \tag{4.35}$$

$$P^- = \frac{L}{2\pi}H, \quad \text{with } H = m^2 \sum_{n=1,2,\ldots} \frac{1}{n} a_n^\dagger a_n \tag{4.36}$$

$K = 1$

We have only one state $\mid 1\rangle \equiv\mid 1^1\rangle$ and $\langle 1 \mid M^2 \mid 1\rangle = m^2$.

K=2

Two states are possible: $\mid 1\rangle \equiv\mid 2^1\rangle$ and $\mid 2\rangle \equiv\mid 1^2\rangle$. We get $\langle 1 \mid M^2 \mid 1\rangle = m^2$ and $\langle 2 \mid M^2 \mid 2\rangle = 4m^2$. In the state $\mid 2\rangle$ we have the longitudinal momentum fractions $x_1 = x_2 = \frac{1}{2}$.

K=4

Possible states are

$$\begin{aligned}
\mid 1\rangle \quad &\equiv\mid 4^1\rangle \quad \text{single particle state} \\
\mid 2\rangle \quad &\equiv\mid 2^2\rangle \quad \text{two particle state} \\
\mid 3\rangle \quad &\equiv\mid 1^1, 3^1\rangle \quad \text{two particle state} \\
\mid 4\rangle \quad &\equiv\mid 1^2, 2^1\rangle \quad \text{three particle state} \\
\mid 5\rangle \quad &\equiv\mid 1^4\rangle \quad \text{four particle state}
\end{aligned} \tag{4.37}$$

In the state $\mid 2\rangle$, the momentum fractions are $\frac{1}{2}$ each. In the state $\mid 3\rangle$ the momentum fractions are $\frac{1}{4}$ and $\frac{3}{4}$.

We again see that K restricts both the maximum number of particles allowed and also the minimum momentum fraction allowed. Obviously the continuum limit is $K \to \infty$.

4.3 Special aspects of gauge theories in DLCQ

So far we have discussed scalar theory in DLCQ. It should be clear that constrained zero mode appears in all interacting theories of interest in DLCQ with periodic boundary condition. In gauge theories other special features occur. We discuss one such feature in this section.

In the continuum formulation of gauge theories on the light-front, it is popular to choose the light-front gauge $A^+ = 0$. It can seen that[7], with periodic boundary condition, we cannot set the zero mode of A^+ to zero; it is indeed gauge invariant.

For simplicity let us consider two dimensional Abelian gauge theory with periodic boundary condition, i.e., $\psi^+(x^+, x^- = -L) = \psi^+(x^+, x^- = +L)$. Consider a gauge transformation

$$A^+(x^+, x^-) \to A'^+(x^+, x^-) = A^+(x^+, x^-) + \partial^+ \chi_1(x^+, x^-) \tag{4.38}$$

with

$$\chi_1(x^+, x^-) = -(1/2) \int_{-L}^{x^-} dz^- A^+(x^+, z^-). \tag{4.39}$$

Then $A'^+ = 0$. However, the new fermion field operator

$$\psi'^+ = e^{ie\chi_1} \psi^+. \tag{4.40}$$

It is easily verified that $\psi'^+(x^+, x^- = -L) = \psi^+(x^+, x^- = -L)$, but

$$\psi'^+(x^+, x^- = L) = e^{-i(e/2) \int_L^{+L} dz^- A^+(x^+, z^-)} \psi^+(x^+, x^- = L). \tag{4.41}$$

Thus $\psi'^+(x^+, x^- = L) \neq \psi'^+(x^+, x^- = -L)$ and periodic boundary condition is lost on ψ'^+.

In order to recover periodic boundary condition on ψ'^+, introduce a second gauge function χ_2:

$$\chi_2(x^+, x^-) = \frac{x^- + L}{4L} \int_{-L}^{+L} dz^- A^+(x^+, z^-). \tag{4.42}$$

Now,

$$\psi''^+(x^+, x^-) = e^{ig(x^- + L)/4L \int_{-L}^{+L} dz^- A^+(x^+, z^-)} e^{-i(g/2) \int_{-L}^{x^-} dz^- A^+(x^+, z^-)} \psi^+(x^+, x^-) \tag{4.43}$$

which has periodic boundary condition. Now, however,

$$\begin{aligned} A''^+(x^+) \quad &= A^+(x^+, x^-) + \partial^+ \chi_1 + \partial^+ \chi_2 \\ &= 0 + 2 \frac{\partial}{\partial x^-} \frac{x^- + L}{4L} \int_{-L}^{+L} dz^- A^+(x^+, z^-) \\ &= \frac{1}{2L} \int_{-L}^{L} dz^- A^+(x^+, z^-) \end{aligned} \tag{4.44}$$

Thus we see that, with periodic boundary conditions, zero mode of A^+ cannot be gauged away.

This can be seen in a simple way. With the gauge transformation $A'^+ = A^+ + \partial^+ \chi$, where χ has periodic boundary condition,

$$\int_{-L}^{+L} dx^- A'^+(x^+, x^-) = \int_{-L}^{+L} dx^- A^+(x^+, x^-). \tag{4.45}$$

4.4 Fourier Series expansions

If $f(x)$ is periodic with period $2L$, we may write

$$f(x) = \frac{a_0}{2} + \sum_{n=1,2,...} \Big[a_n Cos\frac{n\pi x}{L} + b_n Sin\frac{n\pi x}{L}\Big] \tag{4.46}$$

with

$$a_n = \frac{1}{L}\int_{-L}^{+L} dy f(y) Cos\frac{n\pi y}{L} \quad n = 0, 1, 2, \ldots \tag{4.47}$$

$$b_n = \frac{1}{L}\int_{-L}^{+L} dy f(y) Sin\frac{n\pi y}{L} \quad n = 1, 2, 3, \ldots \tag{4.48}$$

For $f(x) = \delta(x)$, we get $a_n = \frac{1}{L}$, $b_n = 0$. Thus we have

$$\delta(x) = \frac{1}{2L}\Big[1 + \sum_{n=1,2,...} [e^{\frac{in\pi x}{L}} + e^{-\frac{in\pi x}{L}}]\Big]. \tag{4.49}$$

For $f(x) = \epsilon(x) = \theta(x) - \theta(-x)$, we get $a_n = 0$, $b_n = \frac{2}{n\pi}[1 - (-1)^n]$. Thus

$$\epsilon(x) = \frac{2}{i\pi} \sum_{n=1,3,...} \frac{1}{n}\Big[e^{\frac{in\pi x}{L}} - e^{-\frac{in\pi x}{L}}\Big]. \tag{4.50}$$

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