APPENDIX B

Further adventures on the light front

B.1 Stability Group

It is well known that the number of Poincare generators that leave a light front $x^+ = 0$ invariant is seven in contrast to six which leave a a space-like surface invariant. Why do we get a larger stability group in front form compared to instant form? An intuitive answer [1] is that " Every Poincare transformation that belongs to the stability group carries a light front into a light front. The manifold of light-like planes is smaller that the manifold of space-like planes. Therefore the image of the plane will coincide with the original one for a larger class of group elements". It is the same reasoning that explains why the stability group is larger for the Galilean group (nine kinematical generators) compared to the Poincare group in instant form.

B.2 Phase space measure

In equal time theory, we have

$$\int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k^0) = \int \frac{d^3k}{2(2\pi)^3\omega_k} \tag{B.1}$$

with $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$.

In light front theory, we have,

$$\begin{aligned} \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2)\theta(k^+) \quad &= \frac{1}{2}\int \frac{dk^- dk^+ d^2k^\perp}{(2\pi)^4 \mid k^+ \mid} 2\pi\delta(k^- - \frac{(k^\perp)^2 + m^2}{k^+})\theta(k^+) \\ &= \int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+}\theta(k^+). \end{aligned} \tag{B.2}$$

B.3 Classical free scalar field

An arbitrary solution to the equation of motion can be written as

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2 - m^2) f(k) e^{-ik\cdot x}. \tag{B.3}$$

Using $d^4k = \frac{1}{2}dk^- dk^+ d^2k^\perp$ and $k^2 = k^+k^- - (k^\perp)^2$, we get

$$\begin{aligned}\phi(x) \quad &= \int_{-\infty}^{0} dk^+ \int d^2k^\perp \frac{1}{2(2\pi)^3 \mid k^+ \mid} f\left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) e^{-\frac{i}{2}\frac{(k^\perp)^2+m^2}{k^+} - \frac{i}{2}k^+x^- + ik^\perp \cdot x^\perp} \\ &+ \int_{0}^{\infty} dk^+ \int d^2k^\perp \frac{1}{2(2\pi)^3 k^+} f\left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) e^{-\frac{i}{2}\frac{(k^\perp)^2+m^2}{k^+} - \frac{i}{2}k^+x^- + ik^\perp \cdot x^\perp}.\end{aligned} \tag{B.4}$$

In the first term change k^+ to $-k^+$. Then first term =

$$\int_{0}^{\infty} dk^+ \int d^2k^\perp \frac{1}{2(2\pi)^3 k^+} f\left(k^- = -\frac{(k^\perp)^2 + m^2}{k^+}, -k^+, -k^\perp\right) e^{\frac{i}{2}\frac{(k^\perp)^2+m^2}{k^+} + \frac{i}{2}k^+x^- - ik^\perp \cdot x^\perp} \tag{B.5}$$

so that we have

$$\begin{aligned}\phi(x) \quad &= \int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+} \theta(k^+)[f\left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) e^{-\frac{i}{2}\frac{(k^\perp)^2+m^2}{k^+} - \frac{i}{2}k^+x^- + ik^\perp \cdot x^\perp} \\ &+ f\left(k^- = -\frac{(k^\perp)^2 + m^2}{k^+}, -k^+, -k^\perp\right) e^{\frac{i}{2}\frac{(k^\perp)^2+m^2}{k^+} + \frac{i}{2}k^+x^- - ik^\perp \cdot x^\perp}].\end{aligned} \tag{B.6}$$

Defining

$$a(k^+, k^\perp) = f\left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) \quad \text{and} \quad a^*(k^+, k^\perp) = f\left(k^- = -\frac{(k^\perp)^2 + m^2}{k^+}, -k^+, -k^\perp\right)$$

we arrive at

$$\phi(x) = \int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+} \theta(k^+)[a(k)e^{-ik.x} + a^*(k)e^{ik.x}]. \tag{B.7}$$

It is interesting to contrast the free field solutions in the instant form and front form. In instant form, the field expansion is

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[a(\mathbf{k})e^{-ik\cdot x} + a^\dagger(\mathbf{k})e^{ik\cdot x}\right] \tag{B.8}$$

with

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \delta^3(\mathbf{k} - \mathbf{k}'). \tag{B.9}$$

Solving for $a(\mathbf{k})$ from Eq. (B.8), we get

$$a(\mathbf{k}) = \int d^3x e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} \left[\omega_k \phi(x) + i \dot{\phi}(x)\right]. \tag{B.10}$$

Thus in instant form we need to know both ϕ and $\dot{\phi}$ to determine $a(\mathbf{k})$ which is a reflection of the fact that $\phi(x)$ obeys a second order differential equation in time and we need to specify both ϕ and $\dot{\phi}$ at $t = 0$ to determine the solution.

In front form, on the other hand, from Eq. (B.7) we get

$$\frac{1}{k^+}a(k) = \int dx^- d^2x^\perp e^{i(\frac{1}{2}k^- x^+ + \frac{1}{2}k^+ x^- - k^\perp \cdot x^\perp)}\phi(x) \tag{B.11}$$

since

$$\int dx^- e^{\frac{i}{2}(k^+ + k'^+)x^-} = 0 \tag{B.12}$$

because of the positivity of plus momenta. Thus to determine $a(k)$, we need to know only ϕ. This reflects the fact that the equation obeyed by ϕ in the front form is first order in "time" [2].

B.4 Free electron theory: Dirac equation and second quantization revisited

Consider the Dirac equation

$$\left[\frac{i}{2}\gamma^+\partial^- + \frac{i}{2}\gamma^-\partial^+ - i\gamma^\perp \cdot \partial^\perp - m\right]\psi = 0. \tag{B.13}$$

$\psi = \psi^+ + \psi^-$ where ψ^- obeys the constraint equation

$$\psi^- = \frac{1}{i\partial^+}[\alpha^\perp \cdot i\partial^\perp + \gamma^0 m]\psi^+, \tag{B.14}$$

and ψ^+ obeys the first order equation of motion

$$i\partial^-\psi^+ = \frac{1}{i\partial^+}[-(\partial^\perp)^2 + m^2]\psi^+. \tag{B.15}$$

Thus we have to solve the equation of motion $[\partial^+\partial^- - (\partial^\perp)^2 + m^2]\psi^+ = 0$.

In the two component representation

$$\psi^+(x) = \begin{pmatrix} \eta(x) \\ 0 \end{pmatrix} \tag{B.16}$$

and we have $[\partial^+\partial^- - (\partial^\perp)^2 + m^2]\eta(x) = 0$ where $\eta(x)$ is a two component object. We can write the positive energy solutions as $\eta(x) = \chi e^{-ip.x}$ where χ has two components and $p.x = \frac{1}{2}\frac{(p^\perp)^2+m^2}{p^+}x^+ + \frac{1}{2}p^+x^- - p^\perp \cdot x^\perp$. We have

$$i\partial^-\psi^+ = p^-\psi^+ = \frac{(p^\perp)^2 + m^2}{p^+}\psi^+ \ \text{ and } \ i\partial^+\psi^+ = p^+\psi^+. \tag{B.17}$$

What about χ? We have, $\Lambda^+\psi^+ = \psi^+$. This implies that $\begin{pmatrix} \chi \\ 0 \end{pmatrix}$ is an eigenstate of Λ^+ with eigenvalue $+1$. Since $\Lambda^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ has two eigenfunctions corresponding to the eigenvalue +1, we can choose

$$\begin{pmatrix} \chi_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \chi_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{B.18}$$

Thus we have two positive energy solutions

$$\begin{pmatrix} \chi_\lambda \\ 0 \end{pmatrix} e^{-\frac{i}{2}p^- x^+ - \frac{i}{2}p^+ x^- + ip^\perp \cdot x^\perp} \tag{B.19}$$

with $p^- = \frac{(p^\perp)^2 + m^2}{p^+}$ and χ_λ is the Pauli spinor with $\lambda = 1, 2$. Note that positive energy solutions necessarily have $p^+ > 0$.

What about negative energy solutions? They are given by

$$\begin{pmatrix} \chi_\lambda \\ 0 \end{pmatrix} e^{\frac{i}{2}p^- x^+ + \frac{i}{2}p^+ x^- - ip^\perp \cdot x^\perp} \tag{B.20}$$

They satisfy

$$i\partial^- \psi^+ = -\frac{(p^\perp)^2 + m^2}{p^+}\psi^+ \quad \text{and} \quad i\partial^+ \psi^+ = -p^+ \psi^+ \tag{B.21}$$

with $p^+ > 0$.

Thus we have generated two positive energy solutions and two negative energy solutions.

How to pass on to the quantum theory?

We can expand the quantum field operator as

$$\begin{aligned} \eta(x) \quad &= \sum_{r=1,2} \chi_r \int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+} \theta(k^+) a_r \left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) e^{-ik_{on}\cdot x} \\ &+ \sum_{r=3,4} \chi_r \int \frac{dk^+ d^2k^\perp}{2(2\pi)^3 k^+} \theta(k^+) a_r \left(k^- = -\frac{(k^\perp)^2 + m^2}{k^+}, -k^+, -k^\perp\right) e^{+ik_{on}\cdot x} \end{aligned} \tag{B.22}$$

We have defined $k_{on} \cdot x = \frac{1}{2}k^-_{on}x^+ + \frac{1}{2}k^+ x^- - k^\perp \cdot x^\perp$ with $k_{on} = \frac{(k^\perp)^2 + m^2}{k^+}$.

In this formalism we have a_1 (a_2) annihilate positive energy, positive momentum, helicity up (down) electrons and a_3 (a_4) annihilate negative energy, negative momentum, helicity up (down) electrons.

As usual we can get rid of all reference to negative energy, negative momentum electrons by introducing positrons with positive energy and positive momentum.

Introduce

$$\sqrt{k^+}\, b_s \left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) = a_r \left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right), \quad s = r = 1, 2 \tag{B.23}$$

and

$$\sqrt{k^+}\, d_s^\dagger \left(k^- = \frac{(k^\perp)^2 + m^2}{k^+}, k^+, k^\perp\right) = a_r \left(k^- = -\frac{(k^\perp)^2 + m^2}{k^+}, -k^+, -k^\perp\right),$$
$$s = 1, r = 4 \text{ and } s = 2, r = 3. \tag{B.24}$$

Thus annihilation of a negative energy electron of momentum $-k^+$, $-k^\perp$ and spin down (up) appears as the creation of a positive energy positron of momentum k^+, $k^\perp$ and spin up (down).

Thus we arrive at the field expansion in quantum theory

$$\eta(x) = \sum_\lambda \chi_\lambda \int \frac{dk^+ d^2k^\perp}{2(2\pi)^2\sqrt{k^+}} \theta(k^+)[b(k,\lambda)e^{-ik\cdot x} + d^\dagger(k,-\lambda)e^{ik\cdot x}]. \tag{B.25}$$

Because of the positivity of k^+, we have,

$$\begin{aligned} b(k,\lambda) &= \sqrt{k^+}\chi_\lambda^\dagger \int dx^- d^2x^\perp e^{ik\cdot x}\eta(x), \\ d^\dagger(k,\lambda) &= \sqrt{k^+}\chi_{-\lambda}^\dagger \int dx^- d^2x^\perp e^{-ik\cdot x}\eta(x). \end{aligned} \tag{B.26}$$

We need to know only the two-component field $\eta(x)$ to determine $b(k,\lambda)$ and $d^\dagger(k,\lambda)$.

It is interesting to compare the situation in instant form. The field operator is given by

$$\psi(x) = \sum_s \int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{E_k}}\left[b(k,s)u(k,s)e^{-ik\cdot x} + d^\dagger(k,s)v(k,s)e^{ik.x}\right] \tag{B.27}$$

with

$$\left\{b(p,s), b^\dagger(p',s')\right\} = \delta_{s,s'}\delta^3(\mathbf{p}-\mathbf{p}'). \tag{B.28}$$

We also have the orthogonality relation

$$u^\dagger(-\mathbf{p},s')v(\mathbf{p},s) = 0. \tag{B.29}$$

Using the orthogonality property given in Eq. (B.29), we get

$$b(k,s) = \frac{1}{\sqrt{(2\pi)^3}}\sqrt{\frac{m}{E_k}}\int d^3x e^{ik\cdot x}\ u^\dagger(k,s)\ \psi(x). \tag{B.30}$$

Thus we need to know all four compoents of $\psi(x)$ to determine $b(k,s)$ since one has to use the orthogonality property of u's and v's.

In the two component representation, let us look at the objects ψ_L^+ and ψ_R^+. We have

$$\psi^+(x) = \begin{pmatrix}\eta(x)\\ 0\end{pmatrix} \tag{B.31}$$

with

$$\eta(x) = \begin{pmatrix}\eta_1(x)\\ \eta_2(x)\end{pmatrix} \tag{B.32}$$

We have $P_R = \frac{1}{2}(1+\gamma^5)$ and $P_L = \frac{1}{2}(1-\gamma^5)$ with

$$\gamma^5 = \begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix} \tag{B.33}$$

Then

$$\psi_R^+ = P_R\psi^+ = \begin{pmatrix} \eta_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{B.34}$$

and

$$\psi_L^+ = P_L\psi^+ = \begin{pmatrix} 0 \\ \eta_2 \\ 0 \\ 0 \end{pmatrix}. \tag{B.35}$$

Thus $\psi_R^+ = P_R\psi^+$ represents a +ve helicity fermion and $\psi_L^+ = P_L\psi^+$ represents a -ve helicity fermion, even when the fermion is *massive.* This makes sense since chirality *is* helicity even for a massive fermion in front form. This is again to be contrasted with the instant form. In that case the right handed and left handed fields defined by $\psi_R = P_R\psi = \frac{1}{2}(1+\gamma^5)\psi$ and $\psi_L = P_L\psi = \frac{1}{2}(1-\gamma^5)\psi$ contain both +ve helicity and -ve helicity states. Only in the massless limit or in the infinite momentum limit, ψ_R becomes the +ve helicity state and ψ_R becomes the -ve helicity state.

B.5 $SL(2,C)$, $E(2) \times D$ and the light front Dirac Equation

Let us look at $SL(2,C)$. We have to consider two types of spinors [3].

B.5.1 Type I

$(\frac{1}{2}, 0) : \vec{J} = \frac{1}{2}\,\vec{\sigma}, \vec{K} = -\frac{i}{2}\,\vec{\sigma}$.

Knowing the explicit forms of $\vec{J}$ and $\vec{K}$ we can construct the light front Dilation operator K^3 and the transverse boosts E^1 and E^2.

$$K^3 = -\frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E^1 = -K^1 + J^2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad E^2 = -K^2 - J^1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{B.36}$$

Let us start with the "rest state" with longitudinal momentum $k^+ = M$ and transverse momentum $0^\perp$. Under the action of the Dilation operator $e^{i\omega K^3}$, $k^+ \to p^+$ where $p^+ = e^\omega k^+ = e^\omega M$. Under the action of the transverse boost $e^{-i\frac{p^\perp \cdot E^\perp}{p^+}}$, $k^\perp \to k^\perp + p^\perp$. Thus

$$\mid p^+, p^\perp\rangle = e^{-iv^\perp \cdot E^\perp} e^{i\omega K^3} \mid M, 0^\perp\rangle \tag{B.37}$$

where $v^\perp = \frac{p^\perp}{p^+}$, $e^\omega = \frac{p^+}{M}$.

A straightforward calculation yields[4]

$$e^{-iv^\perp \cdot E^\perp} e^{i\omega K^3} = \frac{1}{\sqrt{p^+ M}}\begin{pmatrix} p^+ & 0 \\ p^1 + ip^2 & M \end{pmatrix}. \tag{B.38}$$

B.5.2 Type II

$(0, \frac{1}{2}) : \vec{J} = \frac{1}{2}\vec{\sigma}, \vec{K} = \frac{i}{2}\vec{\sigma}$.

Again knowing the explicit forms of $\vec{J}$ and $\vec{K}$ we can construct the light front Dilation operator K^3 and the transverse boosts E^1 and E^2.

$$K^3 = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; E^1 = -K^1 + J^2 = \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \; E^2 = -K^2 - J^1 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{B.39}$$

A straightforward calculation yields

$$e^{-iv^\perp \cdot E^\perp} e^{i\omega K^3} = \frac{1}{\sqrt{p^+ M}}\begin{pmatrix} M & -(p^1 - ip^2) \\ 0 & p^+ \end{pmatrix}. \tag{B.40}$$

From Eqs. (B.38) and (B.40), we indeed verify[5] that in terms of the group $SL(2,C)$ which covers the homogeneous Lorentz group, $E(2) \times D$ which leaves the null plane invariant consists of matrices of the form

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \tag{B.41}$$

which implies that the spinor representation is reducible.

B.5.3 Spinors and the Dirac Equation

Let us denote the 2-spinor in $(\frac{1}{2}, 0)$ by $\phi_{1\lambda}$ and the 2-spinor in $(0, \frac{1}{2})$ by $\phi_{2\lambda}$.
Then

$$\phi_{1\lambda}(p^+, p^\perp) = \frac{1}{\sqrt{p^+ M}}\begin{pmatrix} p^+ & 0 \\ p^1 + ip^2 & M \end{pmatrix} \phi_{1\lambda}(M, 0^\perp), \tag{B.42}$$

and

$$\phi_{2\lambda}(p^+, p^\perp) = \frac{1}{\sqrt{p^+ M}}\begin{pmatrix} M & -(p^1 - ip^2) \\ 0 & p^+ \end{pmatrix} \phi_{2\lambda}(M, 0^\perp), \tag{B.43}$$

with

$$\phi_{1\lambda}(M, 0^\perp) = \phi_{2\lambda}(M, 0^\perp) = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{B.44}$$

From Eqs. (B.42) and (B.43), we get

$$\phi_{1\lambda}(p^+, p^\perp) = \begin{pmatrix} p^+ & 0 \\ p^1 + ip^2 & M \end{pmatrix}\begin{pmatrix} M & -(p^1 - ip^2) \\ 0 & p^+ \end{pmatrix}^{-1} \phi_{2\lambda}(p^+, p^\perp), \tag{B.45}$$

and

$$\phi_{2\lambda}(p^+, p^\perp) = \begin{pmatrix} M & -(p^1 - ip^2) \\ 0 & p^+ \end{pmatrix}\begin{pmatrix} p^+ & 0 \\ p^1 + ip^2 & M \end{pmatrix}^{-1} \phi_{1\lambda}(p^+, p^\perp). \tag{B.46}$$

Using

$$\begin{pmatrix} M & -(p^1 - ip^2) \\ 0 & p^+ \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{M} & \frac{p^1-ip^2}{Mp^+} \\ 0 & \frac{1}{p^+} \end{pmatrix}, \tag{B.47}$$

$$\begin{pmatrix} p^+ & 0 \\ p^1 + ip^2 & M \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{p^+} & 0 \\ \frac{-p^1-ip^2}{Mp^+} & \frac{1}{M} \end{pmatrix}, \tag{B.48}$$

we get

$$\phi_{1\lambda}(p^+, p^\perp) = \frac{1}{M} \begin{pmatrix} p^+ & p^1 - ip^2 \\ p^1 + ip^2 & p^- \end{pmatrix} \phi_{2\lambda}(p^+, p^\perp), \tag{B.49}$$

and

$$\phi_{2\lambda}(p^+, p^\perp) = \frac{1}{M} \begin{pmatrix} p^- & -p^1 + ip^2 \\ -p^1 - ip^2 & p^+ \end{pmatrix} \phi_{1\lambda}(p^+, p^\perp), \tag{B.50}$$

Combining Eqs. (B.49) and (B.50), we can write

$$M\phi_\lambda(p^+, p^\perp) = \left[\begin{pmatrix} 0 & 0 & p^+ & 0 \\ 0 & 0 & 0 & p^- \\ p^- & 0 & 0 & 0 \\ 0 & p^+ & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma^\perp \cdot p^\perp \\ -\sigma^\perp \cdot p^\perp & 0 \end{pmatrix} \right] \phi_\lambda(p^+, p^\perp), \tag{B.51}$$

with

$$\phi_\lambda(p^+, p^\perp) = \begin{pmatrix} \phi_{1\lambda}(p^+, p^\perp) \\ \phi_{2\lambda}(p^+, p^\perp) \end{pmatrix}. \tag{B.52}$$

Recalling the explicit form of Dirac matrices in the Weyl representation [3],

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \gamma^\perp = \begin{pmatrix} 0 & -\sigma^\perp \\ \sigma^\perp & 0 \end{pmatrix}. \tag{B.53}$$

Then

$$\gamma^+ = \gamma^0 + \gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \gamma^- = \gamma^0 - \gamma^3 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}. \tag{B.54}$$

Thus we see that Eq. (B.51) is of the form

$$[\gamma \cdot p - m]\, \phi(p) = 0. \tag{B.55}$$

The explicit form of the four component spinors are

$$\phi_\uparrow(p) = \frac{1}{\sqrt{p^+ M}} \begin{pmatrix} p^+ \\ p^1 + ip^2 \\ M \\ 0 \end{pmatrix} \eta_1, \tag{B.56}$$

$$\phi_\downarrow(p) = \frac{1}{\sqrt{p^+ M}} \begin{pmatrix} 0 \\ M \\ -p^1 + ip^2 \\ p^+ \end{pmatrix} \eta_2. \tag{B.57}$$

The explicit forms of the spinors agree with that given in Ref. [4].

B.5.4 The constraint equation

The spinor

$$\phi = \begin{pmatrix} \phi_{1\uparrow} \\ \phi_{1\downarrow} \\ \phi_{2\uparrow} \\ \phi_{2\downarrow} \end{pmatrix}. \tag{B.58}$$

With

$$\Lambda^{+} = \frac{1}{4}\gamma^{-}\gamma^{+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.59}$$

and

$$\Lambda^{-} = \frac{1}{4}\gamma^{+}\gamma^{-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{B.60}$$

we have

$$\phi^{+} = \Lambda^{+}\phi = \begin{pmatrix} \phi_{1\uparrow} \\ 0 \\ 0 \\ \phi_{2\downarrow} \end{pmatrix} \tag{B.61}$$

and

$$\phi^{-} = \Lambda^{-}\phi = \begin{pmatrix} 0 \\ \phi_{1\downarrow} \\ \phi_{2\uparrow} \\ 0 \end{pmatrix}. \tag{B.62}$$

From Eqs. (B.49) and (B.50) we get

$$\begin{aligned} p^{+}\phi_{1\downarrow} &= (p^{1} + ip^{2})\phi_{1\uparrow} + M\phi_{2\downarrow} \\ p^{+}\phi_{2\uparrow} &= (-p^{1} + ip^{2})\phi_{2\downarrow} + M\phi_{1\uparrow} \end{aligned} \tag{B.63}$$

which shows that two components of ϕ are not dynamical (these two equations do not contain the energy p^{-}). The constraint equation Eq. (B.63) can be cast in the elegant form

$$p^{+}\phi^{-} = \left[\alpha^{\perp} \cdot p^{\perp} + \gamma^{0}M\right]\phi^{+}. \tag{B.64}$$

Thus ϕ^{+} is the dynamical field and ϕ^{-} is the constraint field.

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