APPENDIX A

Remembrance of things past

You will, of course, know the
Lorentz transformation;
you will never forget *that* on a
desert island or anywhere else.

R.P. Feynman

When one first encounters light-front literature, one often sees statements like "boosts are simple" or "seven generators of the Poincare group are kinematical compared to six in the instant form" and the like. One of the purpose of this chapter is to unravel the mystery behind these statements in the context of special relativity so that one may understand them without the heavy load of quantum field theory. Following Parker and Schmieg ([1] and [2]) we also take a cursory look at Maxwell's equations in vacuum in terms of the light-front variables. The electromagnetic field tensor is usually expressed in terms of the six independent components namely three electric and three magnetic field strengths. It is shown that a different set of six field strengths emerge when we study the transformation properties of Maxwell equation in light-front variables. We also remind the reader of the utility of light-front variables to solve free field equations.

A.1 Homogeneous Lorentz Group

In this section we begin by considering the generators of boost and rotation in equal time dynamics. The alert reader will notice that we have borrowed heavily from [3].

First we consider a pure boost along the negative 3-axis. The relationship between space and time of two systems of coordinates, one, $\tilde{S}$, in uniform motion along the negative 3-axis with speed v relative to other, S, is given by

$$\tilde{x}^0 = \gamma(x^0 + \beta\, x^3) \tag{A.1}$$

$$\tilde{x}^3 = \gamma(x^3 + \beta\, x^0) \tag{A.2}$$

where

$$\beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}. \tag{A.3}$$

Introduce the parameter ϕ such that

$$\gamma = cosh\,\phi, \quad \beta\gamma = sinh\,\phi. \tag{A.4}$$

We have

$$\begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix} = B^3 \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \tag{A.5}$$

where the Boost matrix

$$B^3 = \begin{pmatrix} cosh\,\phi & 0 & 0 & sinh\,\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sinh\,\phi & 0 & 0 & cosh\,\phi \end{pmatrix}. \tag{A.6}$$

The generator of this boost transformation is

$$K^3 = \frac{1}{i}\frac{\partial B^3}{\partial \phi}\,|_{\phi=0} = -i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A.7}$$

Similarly

$$K^1 = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{A.8}$$

and

$$K^2 = -i \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.9}$$

Note that the matrices K^i are not hermitian and hence the operators $e^{i\phi K^i}$ are not unitary.

Next consider rotations. For a clockwise rotation of a vector $\mathbf{V}$ about the 3-axis through an angle θ we have

$$\begin{pmatrix} \tilde{V}^1 \\ \tilde{V}^2 \\ \tilde{V}^3 \end{pmatrix} = R^3(\theta) \begin{pmatrix} V^1 \\ V^2 \\ V^3 \end{pmatrix} \tag{A.10}$$

where the rotation matrix

$$R^3 = \begin{pmatrix} cos\,\theta & sin\,\theta & 0 \\ -sin\,\theta & cos\,\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{A.11}$$

The generator J_3 of this rotation is

$$J^3 = \frac{1}{i}\frac{\partial R^3}{\partial \theta}\,|_{\theta=0} = \; -i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.12}$$

In a four-dimensional notation

$$J^3 \;=\; -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.13}$$

Similarly,

$$J^1 \;=\; -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{A.14}$$

and

$$J^2 \;=\; -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{A.15}$$

Note that the matrices J^i are hermitian and hence the operators $e^{i\phi J^i}$ are unitary.

Note that the three generators of rotation J^i leave $x^0 = 0$ invariant whereas the three generators of boost K^i *do not* leave $x^0 = 0$ invariant. Thus three generators of the homogeneous Lorentz group are kinematic and three are dynamic.

One can explicitly verify that

$$[J^i, J^j] = i\epsilon_{ijk} J^k, \tag{A.16}$$

$$[J^i, K^i] = 0, \tag{A.17}$$

$$[J^i, K^j] = i\epsilon_{ijk} K^k, \tag{A.18}$$

$$[K^i, K^j] = -i\epsilon_{ijk} J^k. \tag{A.19}$$

Introduce $M^{\mu\nu}$ with $M^{ij} = \epsilon_{ijk} J^k$ and $M^{0i} = K^i$. Then the commutator algebra of the homogeneous Lorentz group can be summarized as

$$[M^{\mu\nu}, M^{\rho\sigma}] = i[-g^{\mu\rho} M^{\nu\sigma} + g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho} + g^{\nu\rho} M^{\mu\sigma}]. \tag{A.20}$$

Next we motivate light-front dynamics. Recall that under a boost along the 3-axis,

$$\tilde{x}^0 = x^0 \, cosh\,\phi \,+\, x^3 \, sinh\,\phi, \; \tilde{x}^3 = x^0 \, sinh\,\phi \,+\, x^3 \, cosh\,\phi \,. \tag{A.21}$$

Suppose we define

$$x^+ = x^0 + x^3, \; x^- = x^0 - x^3. \tag{A.22}$$

Then

$$\tilde{x}^+ = e^{\phi} \, x^+, \; \tilde{x}^- = e^{-\phi} \, x^- \; . \tag{A.23}$$

Thus boost along 3-axis becomes a scale transformation for the variables $\tilde{x}^+$ and $\tilde{x}^-$! Thus, $x^+ = 0$ is invariant under boost along 3-axis.

Define $E^1 = -K^1 + J^2$. Explicitly

$$E^1 = -i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{A.24}$$

For infinitesimal ϕ,

$$e^{i\phi E^1} = 1 + i\phi E^1 = \begin{pmatrix} 1 & -\phi & 0 & 0 \\ -\phi & 1 & 0 & -\phi \\ 0 & 0 & 1 & 0 \\ 0 & \phi & 0 & 1 \end{pmatrix}. \tag{A.25}$$

Under this infinitesimal transformation

$$\begin{aligned} \tilde{x}^0 &= x^0 - \phi x^1, \\ \tilde{x}^1 &= -\phi x^0 + x^1 - \phi x^3 = x^1 - \phi x^+, \\ \tilde{x}^2 &= x^2, \\ \tilde{x}^3 &= \phi x^1 + x^3. \end{aligned} \tag{A.26}$$

Thus

$$\tilde{x}^+ = x^+. \tag{A.27}$$

Notice that the action of E^1 is just like a Galilean boost in non-relativistic mechanics.

Define $F^1 = -K^1 - J^2$. Explicitly

$$F^1 = -i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{A.28}$$

For infinitesimal ϕ,

$$e^{i\phi F^1} = 1 + i\phi F^1 = \begin{pmatrix} 1 & -\phi & 0 & 0 \\ -\phi & 1 & 0 & \phi \\ 0 & 0 & 1 & 0 \\ 0 & -\phi & 0 & 1 \end{pmatrix}. \tag{A.29}$$

Under this infinitesimal transformation

$$\begin{aligned} \tilde{x}^0 &= x^0 - \phi x^1, \\ \tilde{x}^1 &= -\phi x^0 + x^1 + \phi x^3 = x^1 - \phi x^-, \\ \tilde{x}^2 &= x^2, \\ \tilde{x}^3 &= -\phi x^1 + x^3. \end{aligned} \tag{A.30}$$

Thus

$$\tilde{x}^+ = x^+ - 2\phi x^1. \tag{A.31}$$

Define $E^2 = -K^2 - J^1$. Explicitly

$$E^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.32}$$

For infinitesimal ϕ,

$$e^{i\phi E^2} = 1 + i\phi E^2 = \begin{pmatrix} 1 & 0 & -\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\phi & 0 & 1 & -\phi \\ 0 & 0 & \phi & 1 \end{pmatrix}. \tag{A.33}$$

Under this infinitesimal transformation

$$\begin{aligned} \tilde{x}^0 &= x^0 - \phi x^2, \\ \tilde{x}^1 &= x^1, \\ \tilde{x}^2 &= -\phi x^0 + x^2 - \phi x^3 = x^2 - \phi x^+, \\ \tilde{x}^3 &= \phi x^2 + x^3. \end{aligned} \tag{A.34}$$

Thus

$$\tilde{x}^+ = x^+. \tag{A.35}$$

Thus the action of E^2 is also just like a Galilean boost in non-relativistic mechanics.

Define $F^2 = -K^2 + J^1$. Explicitly

$$F^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{A.36}$$

For infinitesimal ϕ,

$$e^{i\phi F^2} = 1 + i\phi F^2 = \begin{pmatrix} 1 & 0 & -\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\phi & 0 & 1 & \phi \\ 0 & 0 & -\phi & 1 \end{pmatrix}. \tag{A.37}$$

Under this infinitesimal transformation

$$\begin{aligned} \tilde{x}^0 &= x^0 - \phi x^2, \\ \tilde{x}^1 &= x^1, \\ \tilde{x}^2 &= x^2 - \phi x^0 + \phi x^3 = x^2 - \phi x^-, \\ \tilde{x}^3 &= -\phi x^2 + x^3. \end{aligned} \tag{A.38}$$

Thus

$$\tilde{x}^+ = x^+ - 2\phi x^2. \tag{A.39}$$

For rotation about the 3-axis, we have, for infinitesimal ϕ,

$$e^{i\phi J^3} = 1 + i\phi J^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\phi & 0 \\ 0 & \phi & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.40}$$

Under this infinitesimal transformation

$$\begin{aligned} \tilde{x}^0 &= x^0, \\ \tilde{x}^1 &= x^1 + \phi x^2, \\ \tilde{x}^2 &= x^2 - \phi x^1, \\ \tilde{x}^3 &= x^3. \end{aligned} \tag{A.41}$$

Thus

$$\tilde{x}^+ = x^+. \tag{A.42}$$

Thus we see that among the six generators of the homogeneous Lorentz group the four generators K^3, J^3, E^1, and E^2 leave $x^+ = 0$ invariant while the two generators F^1 and F^2 *do not* leave $x^+ = 0$ invariant.

One can explicitly verify that

$$[F^1, F^2] = 0,\ [J^3, F^i] = i\epsilon^{ij} F^j \quad i,j = 1,2 \tag{A.43}$$

with $\epsilon^{12} = -\epsilon^{21} = 1,\ \epsilon^{11} = \epsilon^{22} = 0$. Thus J^3 , F^1 , and F^2 form a closed algebra.

Also

$$[E^1, E^2] = 0,\ [K^3, E^i] = iE^i. \tag{A.44}$$

Thus K^3, E^1, and E^2 also form a closed algebra.

A.2 Maxwell's Equations in Vacuum

The presentation in this section closely follows [1].

The antisymmetric tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{A.45}$$

has six independent components. In instant form we choose them to be

$$F^{0i} = -E^i, \text{ and } F^{ij} = -\epsilon^{ijk} B^k. \tag{A.46}$$

Two of the Maxwell equations

$$\vec{\nabla}.\vec{E} = 0, \quad \vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} \tag{A.47}$$

are summarized in the equation

$$\partial_\mu F^{\mu\nu} = 0. \tag{A.48}$$

The other two Maxwell equations

$$\vec{\nabla}.\vec{B} = 0, \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{A.49}$$

follow from the definitions Eqs. (A.45) and (A.46). Note that the set in Eq.(A.49) can be obtained from the set in Eq.(A.47) by the interchange

$$\vec{E} \to \vec{B}, \quad \vec{B} \to -\vec{E}. \tag{A.50}$$

A.2.1 Transformation properties of the fields

Introduce the light-front variables

$$x^+ = x^0 + x^3 = t + z, \quad x^- = x^0 - x^3 = t - z. \tag{A.51}$$

We also use the notation $(x^1, x^2) = (x, y)$. Now,

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x^+} - \frac{\partial}{\partial x^-}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-}\ . \tag{A.52}$$

From the set given in Eq. (A.49) we get

$$\frac{\partial E^3}{\partial x} + \frac{\partial}{\partial x^-}[E^1 - B^2] - \frac{\partial}{\partial x^+}[E^1 + B^2] = 0, \tag{A.53}$$

$$\frac{\partial E^3}{\partial y} + \frac{\partial}{\partial x^-}[E^2 + B^1] - \frac{\partial}{\partial x^+}[E^2 - B^1] = 0, \tag{A.54}$$

$$\frac{\partial}{\partial y}[E^1 - B^2] - \frac{\partial}{\partial x}[E^2 + B^1] - 2\frac{\partial}{\partial x^+}B^3 = 0. \tag{A.55}$$

Under a Lorentz boost along the negative z axis we have

$$\tilde{x}^+ = e^{\phi}x^+, \quad \tilde{x}^- = e^{-\phi}x^-. \tag{A.56}$$

Using

$$\frac{\partial}{\partial \tilde{x}^+} = e^{-\phi}\frac{\partial}{\partial x^+}, \frac{\partial}{\partial \tilde{x}^-} = e^{\phi}\frac{\partial}{\partial x^-}, \frac{\partial}{\partial \tilde{x}} = \frac{\partial}{\partial x}, \frac{\partial}{\partial \tilde{y}} = \frac{\partial}{\partial y}, \tag{A.57}$$

we have

$$\frac{\partial E^3}{\partial \tilde{x}} + e^{-\phi}\frac{\partial}{\partial \tilde{x}^-}[E^1 - B^2] - e^{\phi}\frac{\partial}{\partial \tilde{x}^+}[E^1 + B^2] = 0, \tag{A.58}$$

$$\frac{\partial E^3}{\partial \tilde{y}} + e^{-\phi}\frac{\partial}{\partial \tilde{x}^-}[E^2 + B^1] - e^{\phi}\frac{\partial}{\partial \tilde{x}^+}[E^2 - B^1] = 0, \tag{A.59}$$

$$\frac{\partial}{\partial \tilde{y}}[E^1 - B^2] - \frac{\partial}{\partial \tilde{x}}[E^2 + B^1] - 2e^{\phi}\frac{\partial}{\partial \tilde{x}^+}B^3 = 0. \tag{A.60}$$

If we require form invariance we should have,

$$\begin{aligned}
\tilde{E}^3 \quad &= E^3, \\
\tilde{B}^3 \quad &= B^3, \\
\tilde{E}^1 - \tilde{B}^2 \quad &= e^{-\phi}[E^1 - B^2], \\
\tilde{E}^1 + \tilde{B}^2 \quad &= e^{\phi}[E^1 + B^2], \\
\tilde{E}^2 + \tilde{B}^1 \quad &= e^{-\phi}[E^2 + B^1], \\
\tilde{E}^2 - \tilde{B}^1 \quad &= e^{\phi}[E^2 - B^1].
\end{aligned} \tag{A.61}$$

Thus form invariance is satisfied if the correct transformation equations of the Maxwell fields hold.

From Eq.(A.61) it follows that

$$\begin{aligned}
\tilde{E}^3 &= E^3, \\
\tilde{B}^3 &= B^3, \\
\tilde{E}^1 &= E^1\, cosh\,\phi + B^2\, sinh\,\phi, \\
\tilde{E}^2 &= E^2\, cosh\,\phi - B^1\, sinh\,\phi, \\
\tilde{B}^1 &= B^1\, cosh\,\phi - E^2\, sinh\,\phi, \\
\tilde{B}^2 &= B^2\, cosh\,\phi + E^1\, sinh\,\phi,
\end{aligned} \tag{A.62}$$

which is the form of the transformation law for the Maxwell fields familiar from equal time dynamics. So far we checked only half the set of Maxwell equations. By making the substitution of Eq.(A.50) the set of equations in Eq. (A.62) are interchanged among themselves.

We can also choose the six independent components of $F^{\mu\nu}$ to be

$$F^{+-} = -2E^3,\ F^{12} = B^3,\ F^{+1} = E^1 + B^2,\ F^{+2} = E^2 - B^1, \\ F^{-1} = E^1 - B^2,\ F^{-2} = E^2 + B^1 \tag{A.63}$$

We note that the six components $F^{+-},\ F^{12},\ F^{+1},\ F^{+2},\ F^{-1},\ F^{-2}$ have simple transformation properties under Lorentz boost along the z axis compared to the components $E^i,\ B^i$.

A.2.2 Plane wave solutions

We seek solutions of Maxwell's equations in vacuum which are independent of x and y. It follows that E^3 and B_z are constants which we put equal to zero. We have

$$\begin{aligned}
\frac{\partial}{\partial x^-}[E^1 - B^2] &= \frac{\partial}{\partial x^+}[E^1 + B^2], \\
\frac{\partial}{\partial x^-}[E^2 + B^1] &= \frac{\partial}{\partial x^+}[E^2 - B^1].
\end{aligned} \tag{A.64}$$

These equations are clearly satisfied if we set

$$\begin{aligned}
E^1 - B^2 &= f_1(x^+), \\
E^1 + B^2 &= f_2(x^-), \\
E^2 + B^1 &= g_1(x^+), \\
E^2 - B^1 &= g_2(x^-).
\end{aligned} \tag{A.65}$$

$$\begin{aligned}
\text{Thus}\quad 2E^1 &= f_1(x^+) + f^2(x^-), \\
2B^2 &= f_2(x^-) - f_1(x^+), \\
2E^2 &= g_1(x^+) + g_2(x^-), \\
2B^1 &= g_1(x^+) - g_2(x^-).
\end{aligned} \tag{A.66}$$

It is interesting to compare the set Eq. (A.66) with eq.(20.25) of Feynman Lectures[4].

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