Lagrange Principle and Boltzmann Distribution of Wealth

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Geoffrey M. Willis, S. Elmsall, W. Yorkshire, UK

Coworkers: Mario Hillebrand, Christian Denk, Thorsten Fründ, Stefan Kallerhoff

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Introduction
Lagrange Principle in Many Particle Systems
Interpretation of the Lagrange Principle
Application of the Lagrange Principle

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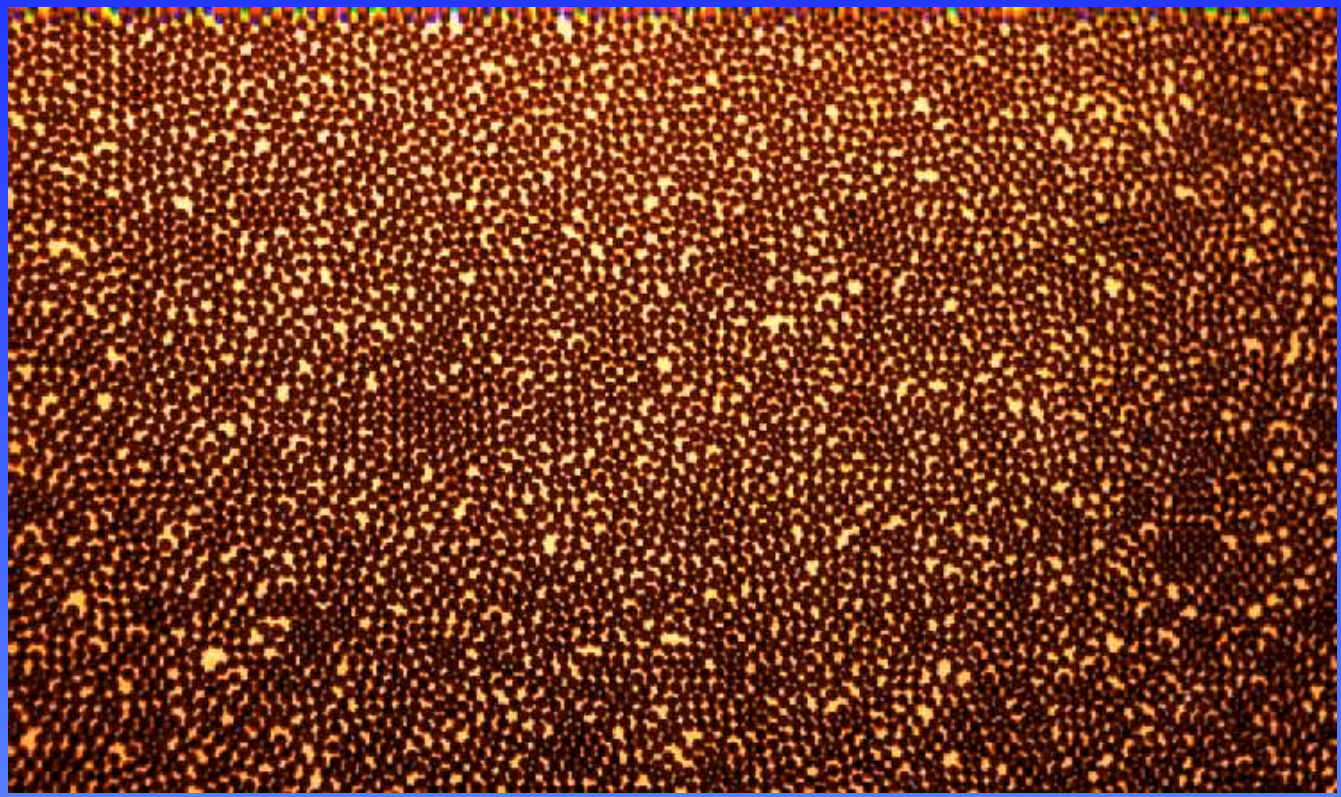
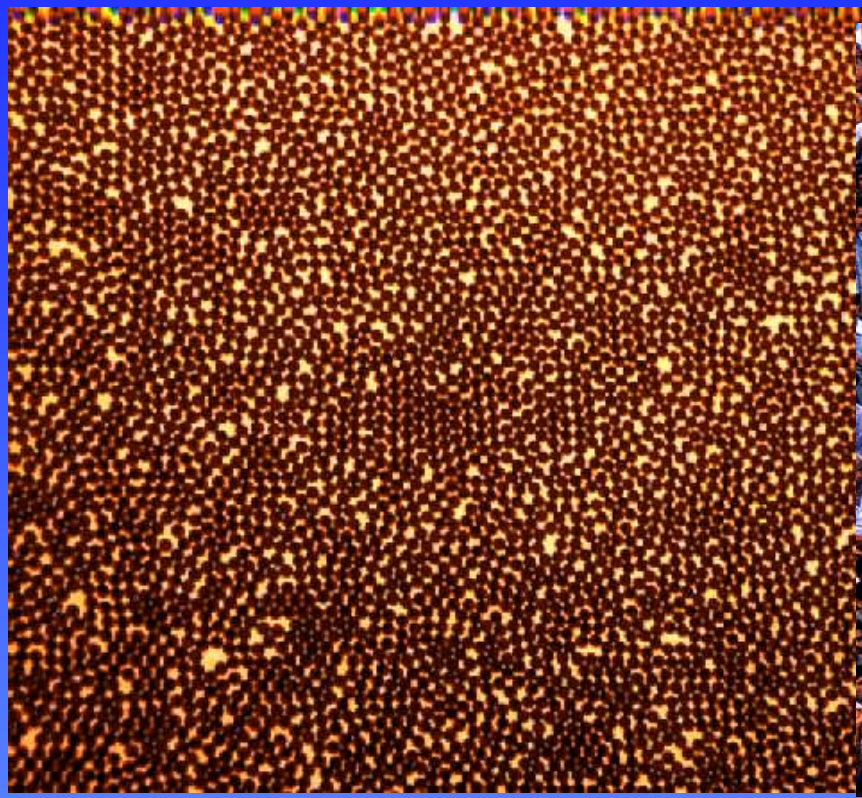
Introduction

Lagrange Principle and Boltzmann Distribution of Wealth



Lagrange principle = statistics with constraints = Gibbs free energy

Lagrange Principle and Boltzmann Distribution of Wealth



Lagrange principle = statistics with constraints = Gibbs free energy

Lagrange Principle and Boltzmann Distribution of Wealth

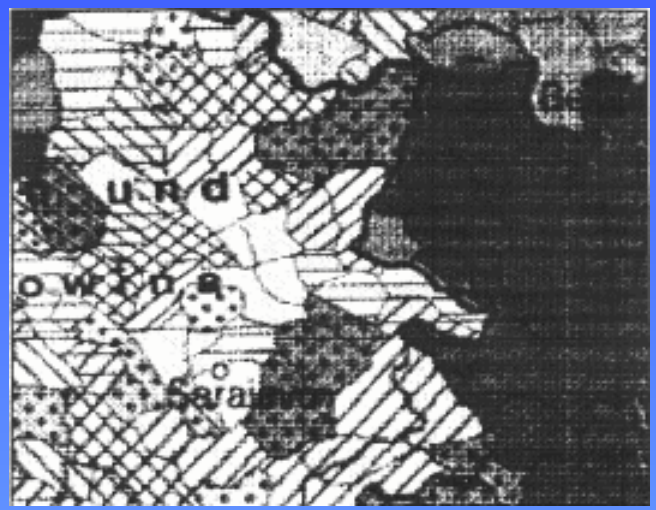
Surface of brass (Cu Zn)

Binary system of copper and zinc

Lagrange Principle and Boltzmann Distribution of Wealth

Surface of brass (Cu Zn)

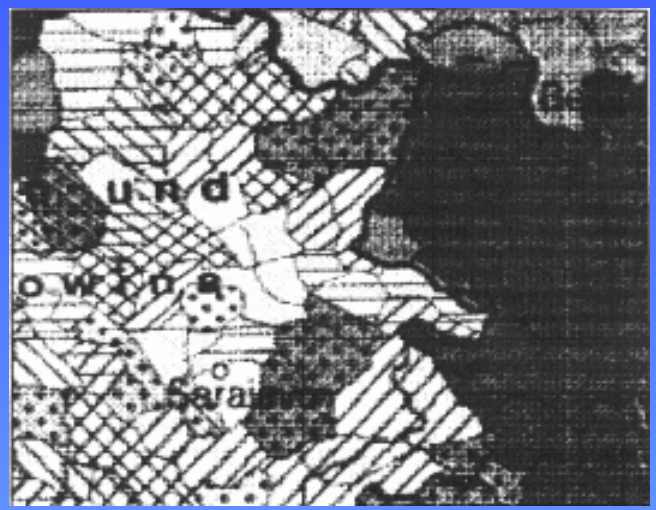
Binary system of copper and zinc

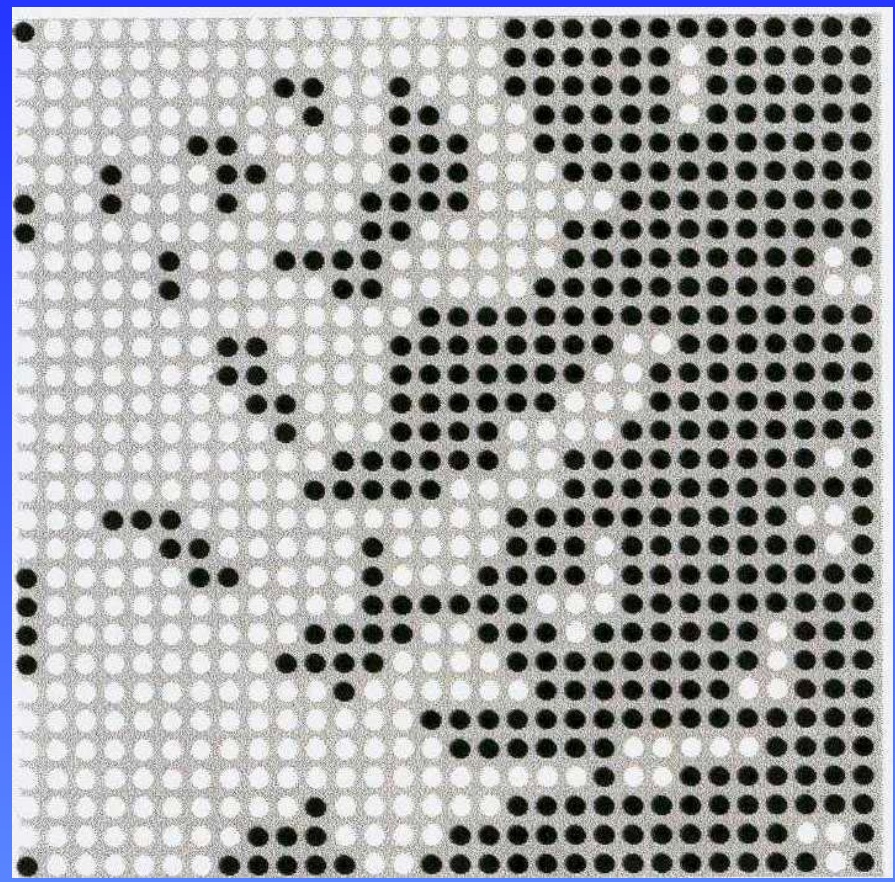


Map of Bosnia 1991

Binary system of Serbs and Non Serbs (Bosnians, Croats)

Lagrange Principle and Boltzmann Distribution of Wealth

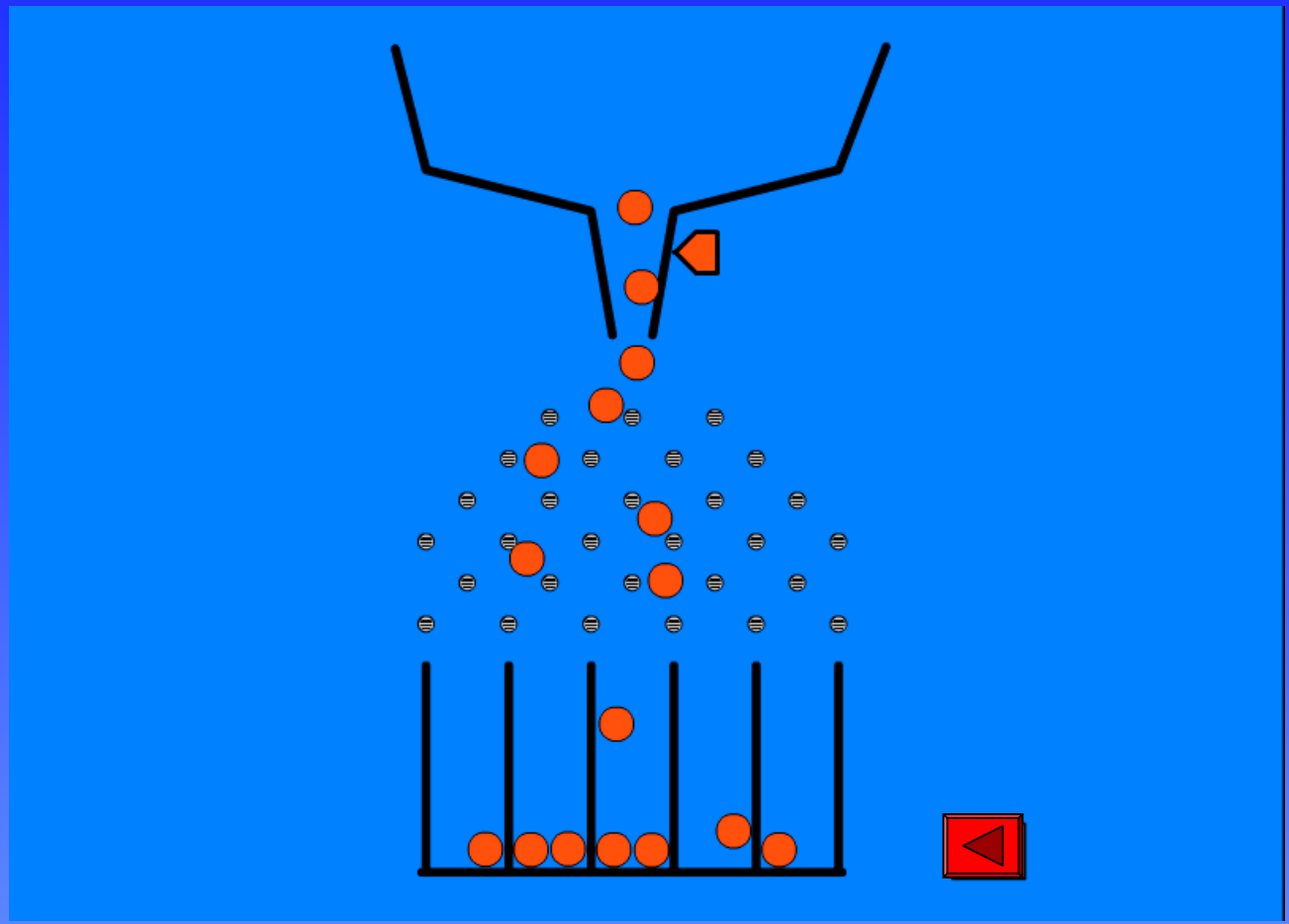




Simulation

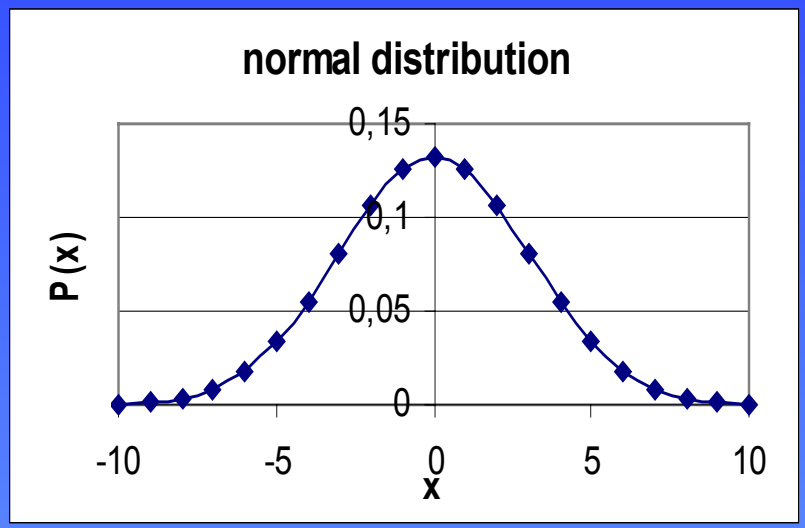
Lagrange Principle in Many Particle Systems

Lagrange Principle and Boltzmann Distribution of Wealth



Mathematics of stochastic systems

Normal distribution



$$P(l, r) = N! / (N_l ! N_r ! 2^N) \rightarrow \max!$$

$$P(x) = (1/\sqrt{(2\pi)}\,\sigma)\, e^{-x^2/2\sigma^2}$$

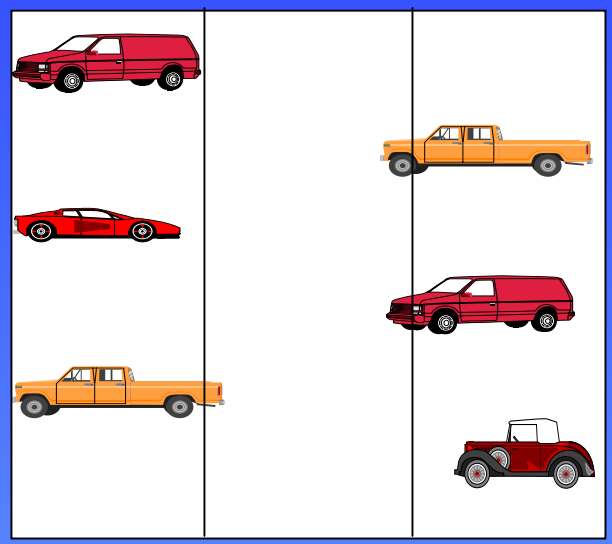
$$x = \tfrac{1}{2}(N_r - N_l)$$

$$\sigma = 1/\sqrt{N}$$

Lagrange Principle and Boltzmann Distribution of Wealth

Probability

$$P(N_l; N_r) = \frac{N!}{N_l! N_r!} \frac{1}{2^N}$$



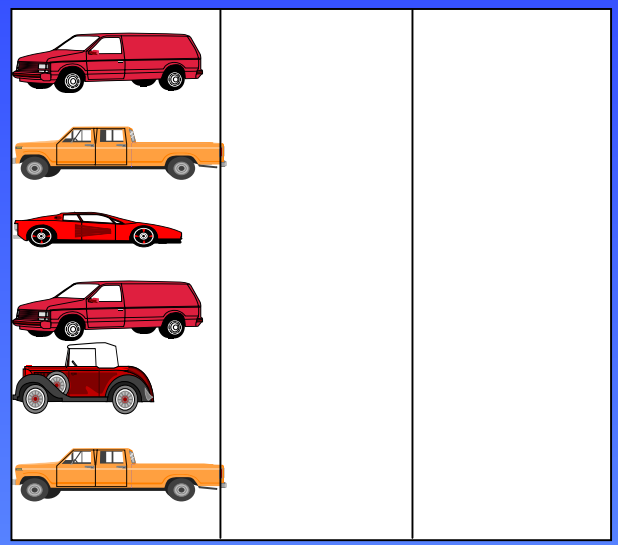
P = 31,25 %

most probable distribution

Lagrange Principle and Boltzmann Distribution of Wealth

Probability

$$P(N_l; N_r) = \frac{N!}{N_l! N_r!} \frac{1}{2^N}$$



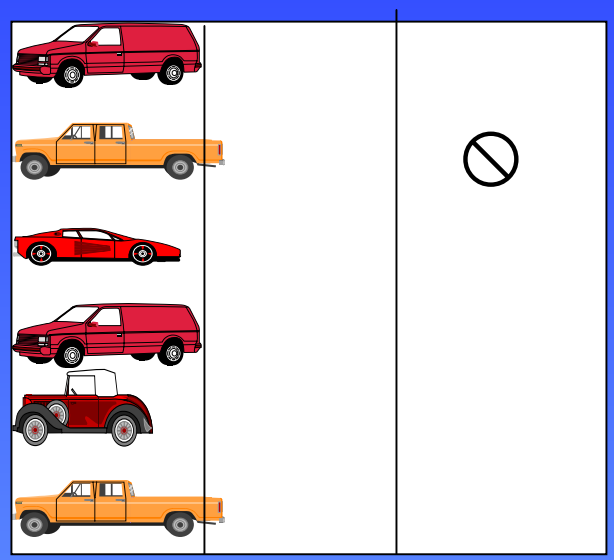
P = 1,56 %

most improbable distribution

Lagrange Principle and Boltzmann Distribution of Wealth

Probability with constraints

$P = ?$



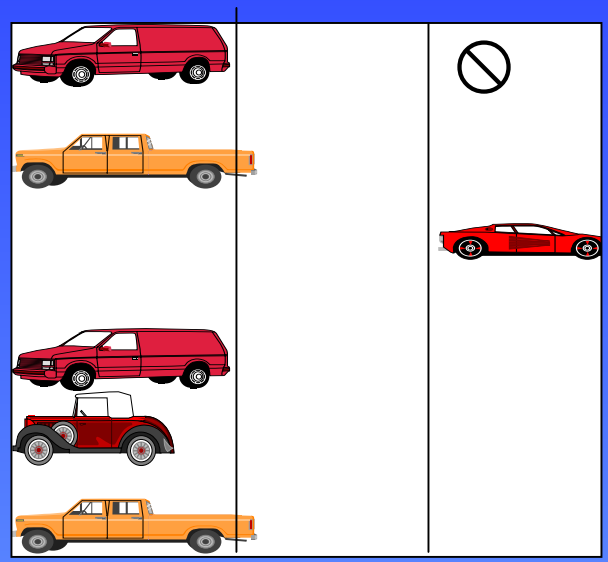
$P = 100\ \% ?$

most probable distribution

Lagrange Principle and Boltzmann Distribution of Wealth

Probability with constraints

$P = ?$

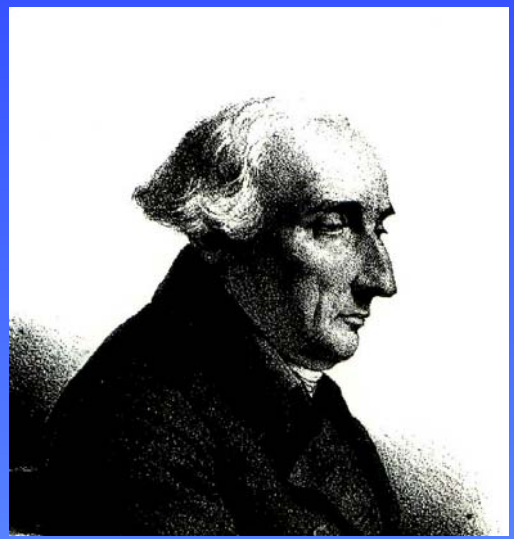


defect distribution

Lagrange Principle and Boltzmann Distribution of Wealth

Probability with constraints (Lagrange Principle)

$L = E + T * \ln P \quad \rightarrow$ maximum!



Joseph de Lagrange

(1736 – 1813)

L : Lagrange function

P : Probability

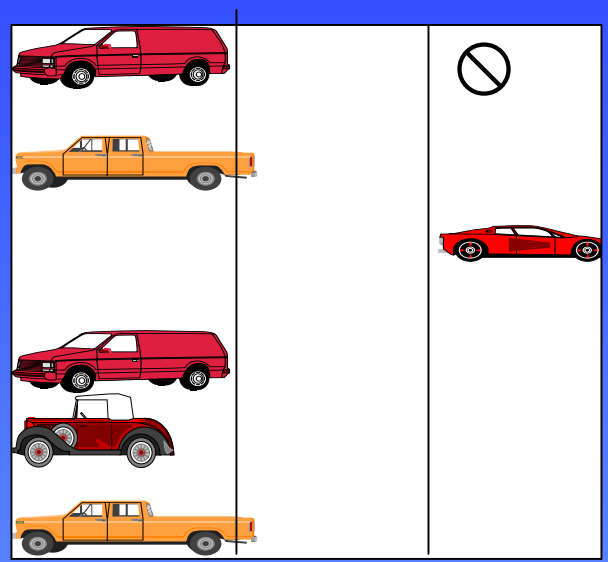
E : constraint: (⊘), fine

T : Lagrange parameter

Lagrange Principle and Boltzmann Distribution of Wealth

Probability with constraints

$$L = E + T * \ln P \quad \rightarrow \text{ maximum!}$$



(Boltzmann)

Equilibrium:

$$L = N_r E + T \ln (N! / N_l! N_r!) = \max$$

$$x_r = N_r / N \ (= 1/6)$$

$$L = -N\{x_r E + T(x_l \ln x_l + x_r \ln x_r)\} = \max$$

$$\partial L / \partial x_r = -N T \{ \ln x_r + 1 + E/T\} = 0$$

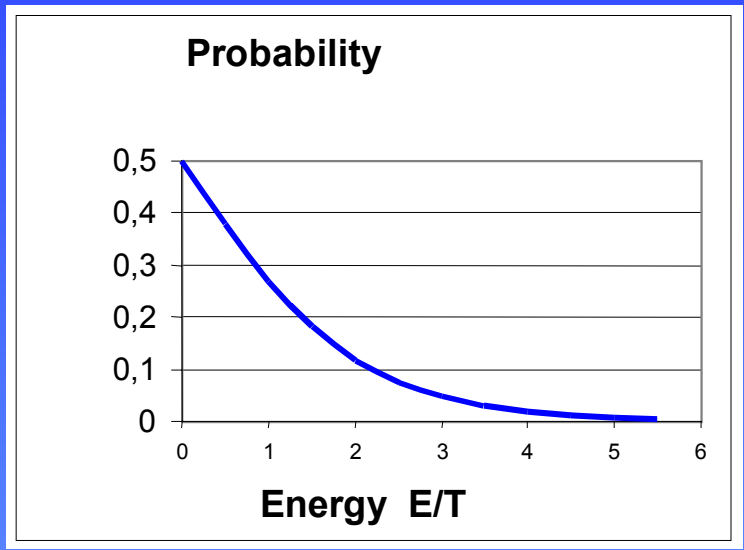
$$T \{ \ln x_r + 1 + E/T\} = 0$$

$$x_r = A \exp\{ - E / T\}$$

Lagrange Principle and Boltzmann Distribution of Wealth

Probability with constraints

$$x_r = A \exp(-E / T)$$

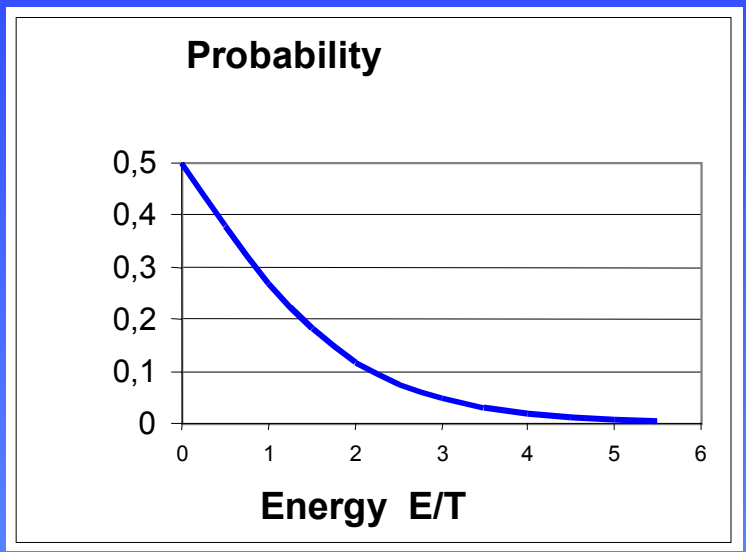


Boltzmann distribution
of defects

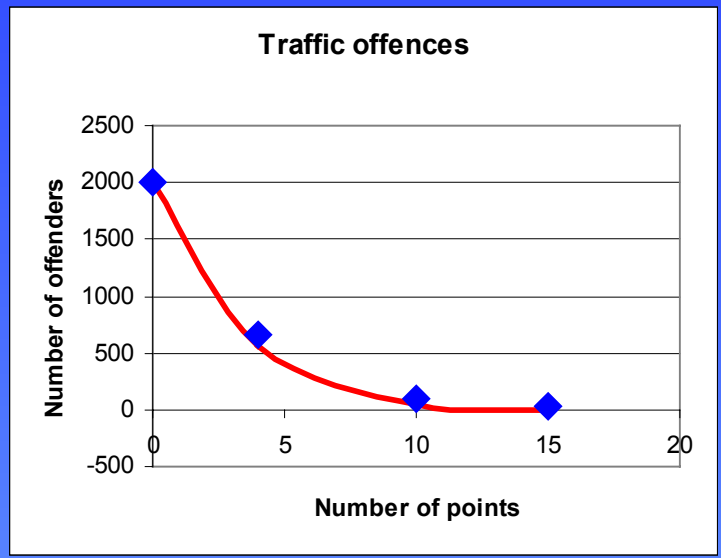
Lagrange Principle and Boltzmann Distribution of Wealth

Probability with constraints

$$x_r = A \exp(-E / T)$$



Boltzmann distribution of defects

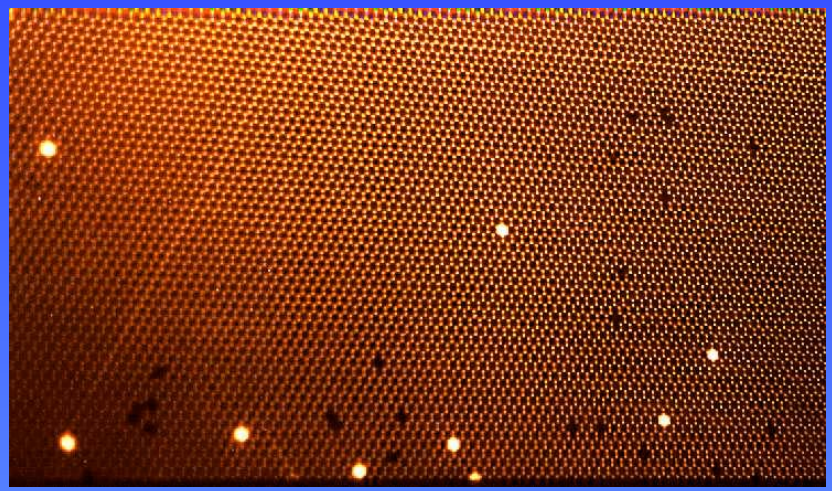


Distribution of traffic violations German Traffic Department 2000

Application of the Lagrange Principle

Natural science

$$L = E(\text{order}) + T \bullet \ln P(\text{disorder}) \rightarrow \text{maximum!}$$



T small : order (solid)

T large : disorder (liquid)

Lagrange Principle and Boltzmann Distribution of Wealth

Society

$$L = E(\text{order}) + T \bullet \ln P(\text{disorder}) \rightarrow \text{maximum!}$$

T small : Order (improbable)

T large : disorder (probable)

Lagrange Principle and Boltzmann Distribution of Wealth

Literature, information:

$$L = E\,(\text{order}) + T \bullet \ln P / P_E\,(\text{disorder}) \rightarrow \text{maximum!}$$

Ben Jonson (1572–1637)
Drink to me only with thine eyes
and I will pledge with mine,
or leave a kiss but on the cup
and I'll not ask for wine.

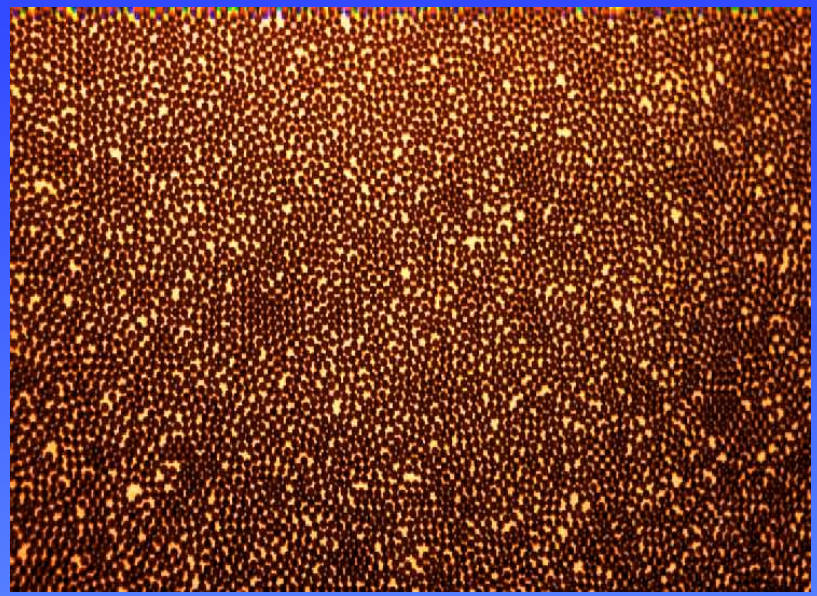
Be njo nson(1 57 2–1 6 37)
D rinktom eon lywi thth ineeyes
an dDw illpl edgew ithmi ne,
orle awea ki ssbu tont hecup
a ndD'l lno task forwi ne.

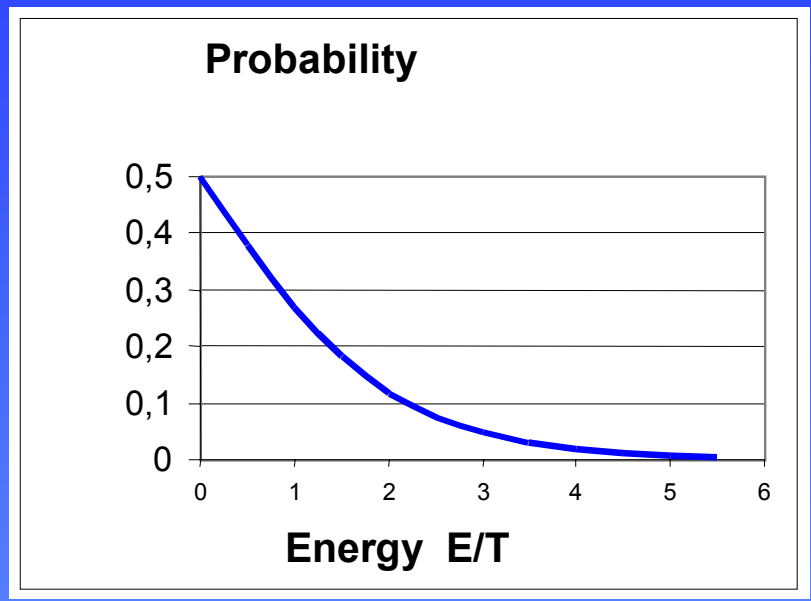
T small : order (correct)

T large: chaos (incorrect)

Lagrange Principle and Boltzmann Distribution of Wealth

Distribution of energy: $x = A \exp(-E / T)$



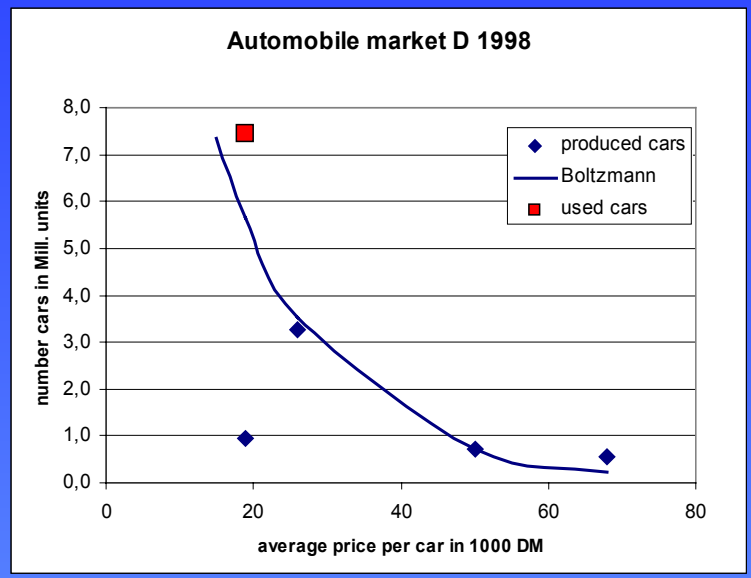


Distribution of energy in gases:

Number of atoms per energy class

Lagrange Principle and Boltzmann Distribution of Wealth

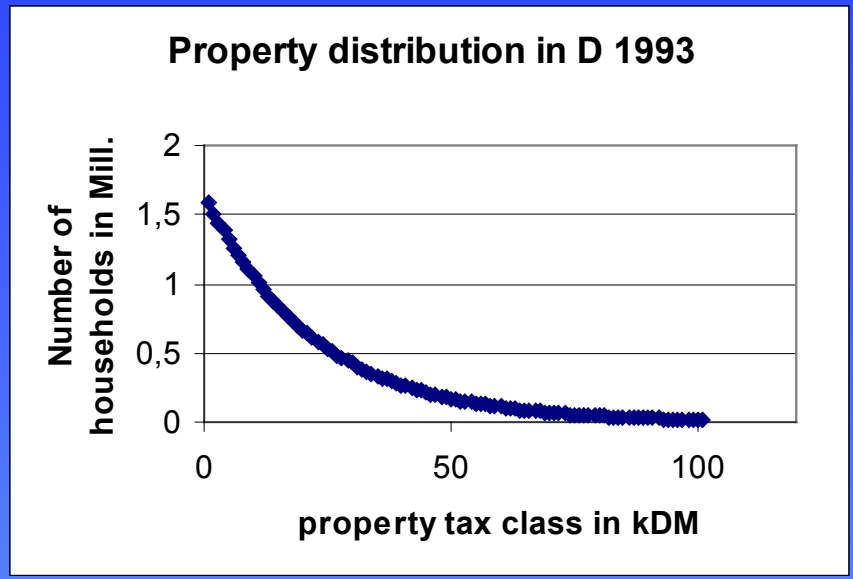
Distribution of prices: $x = A \exp(-E / T)$



Distribution of prices in free markets: Number of cars per tax class

Lagrange Principle and Boltzmann Distribution of Wealth

Distribution of Wealth: $x = A \exp(-E / T)$



Distribution of Wealth in free societies: Number of households per property class

Lagrange Principle and Boltzmann Distribution of Wealth

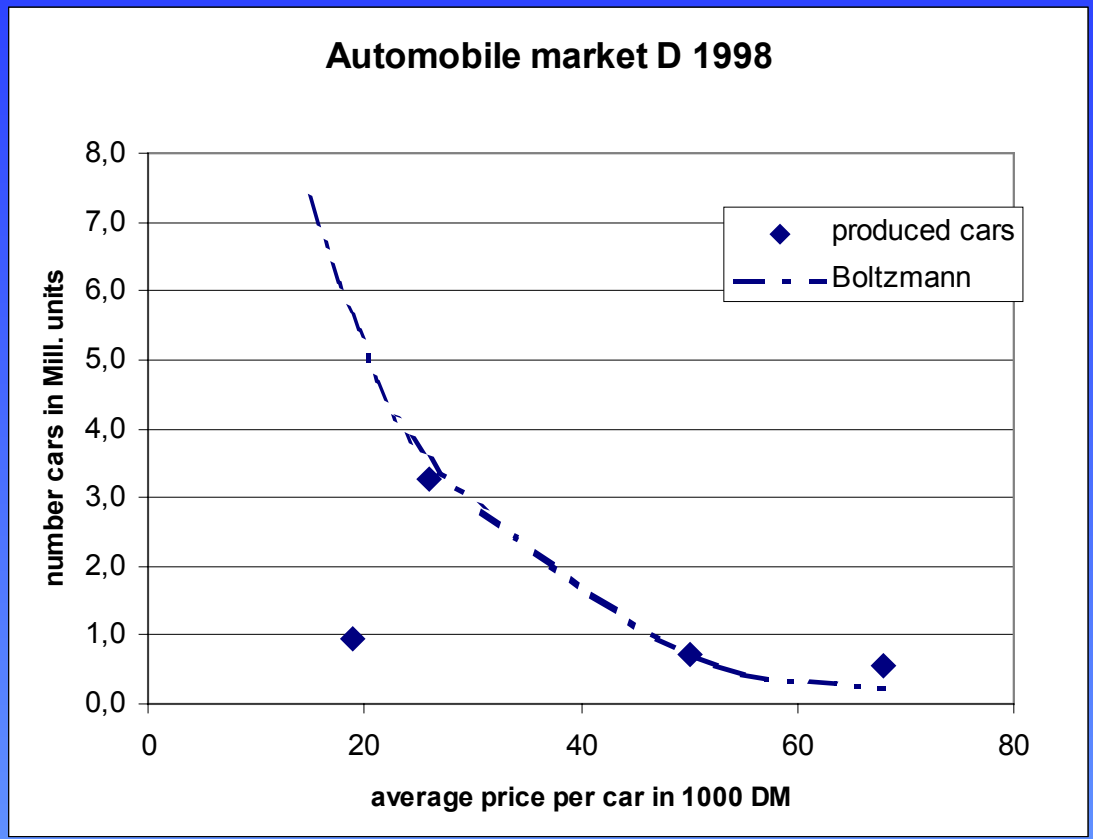
German automobile market 1998

$x_s = A \exp(-P / T)$?

Produced cars 1998

price in kDM	units in 1000
19	940
26	3250
50	720
68	540
	5,45 Mill.

Boltzmann distribution?



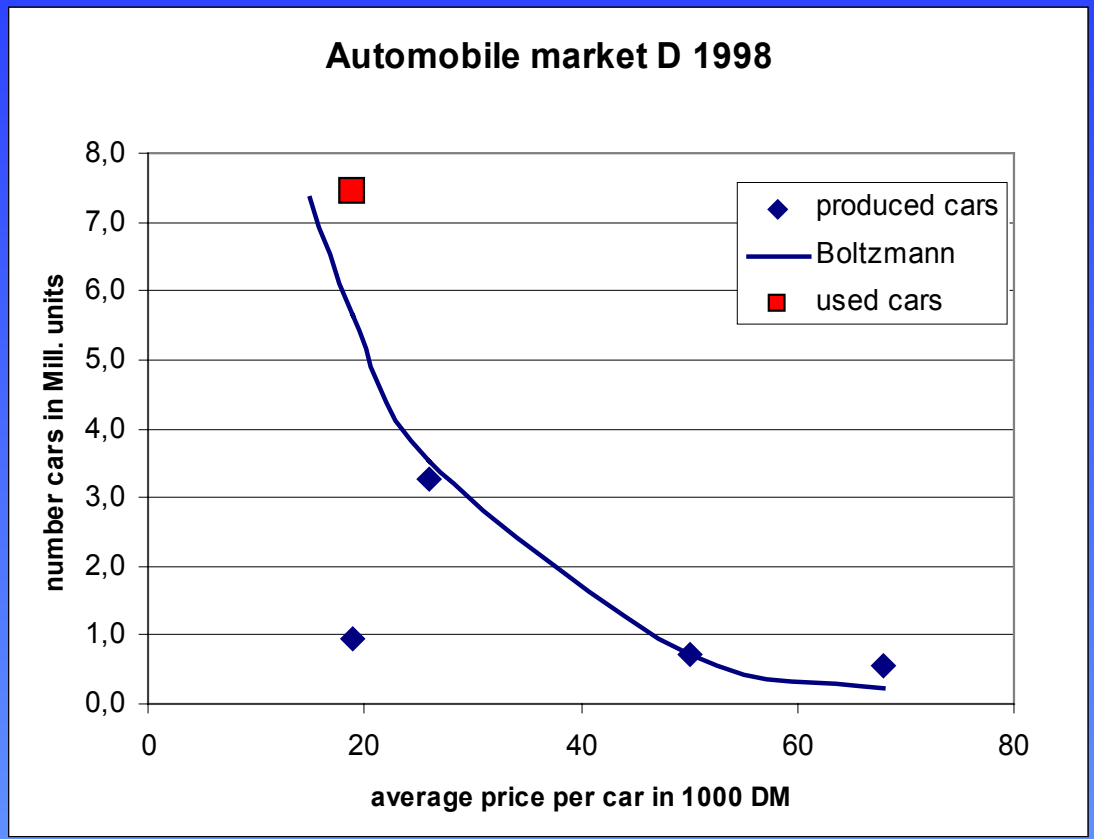
German automobile market 1998 $x_s = A \exp(-P / T)$?

Produced cars 1998

price in kDM	units in 1000
19	940
26	3250
50	720
68	540
	5,45 Mill.

Flensburg:

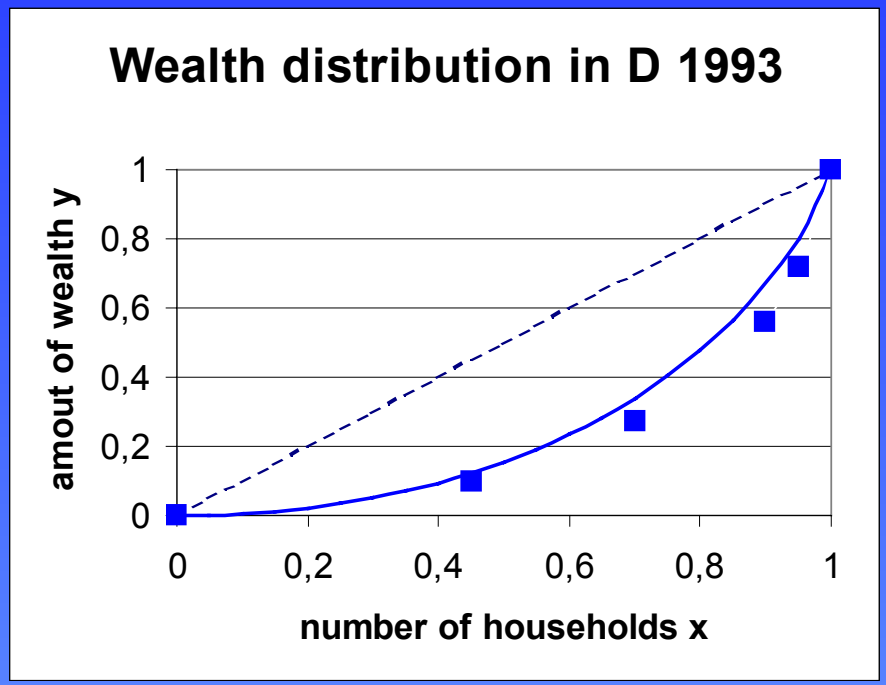
new	3,75 Mill.
used	7,45 Mill
total	11,20 Mill.



Lagrange Principle and Boltzmann Distribution of Wealth

Lagrange Principle and Boltzmann Distribution of Wealth

Lorenz distribution of wealth in Germany 1993 (DIW)



Property distribution in Germany 1993 (DIW estimated)

Total property	9920	Bill. DM
Number of households	35,6	Mill.
Mean property per household	278	kDM / Hh

Property class x in kDM	Number N of Hh in %	Property K of Hh in %
=0	1,5	0
<100	44,5	9,5
100-250	24,7	17,6
250-500	20,3	28,1
500-1000	6,3	16,8
>1000	2,7	28

$$y_{\text{Boltzmann}} = x + (1-x)\ln(1-x)$$

Lagrange Principle and Boltzmann Distribution of Wealth

Distribution of wealth in Germany 1993 (DIW)

N(k): number of Hh in class k

K(k): amount of capital in class k

$$y(a) = \int_0^a K(k)dk \,/\, \int_0^\infty K(k)dk$$

$$x(a) = \int_0^a N(k)dk \,/\, \int_0^\infty N(k)dk$$

$N(k) = A \exp(-k / T)$ Boltzmann distribution

$y(x) = x + (1 - x) \ln (1 - x)$

Boltzmann distribution of wealth for Germany 1993 (DIW)

$N(k) = A \exp(- k / T)$ number of Hh in class k

$K(k) = A\, k \exp(- k / T)$ amount of capital in class k

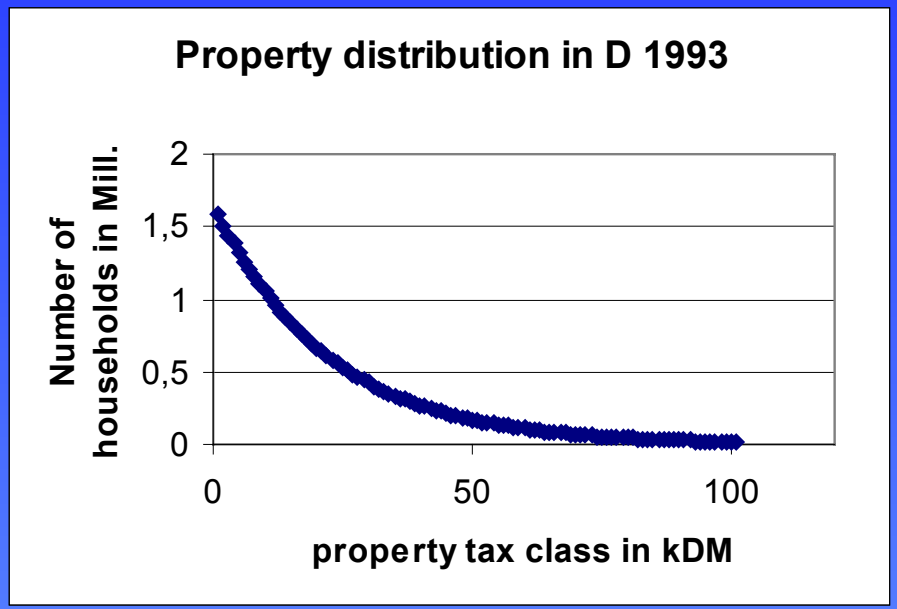
$N = \int N(k)\, dk = A\, T$ = 36 Mill. Hh

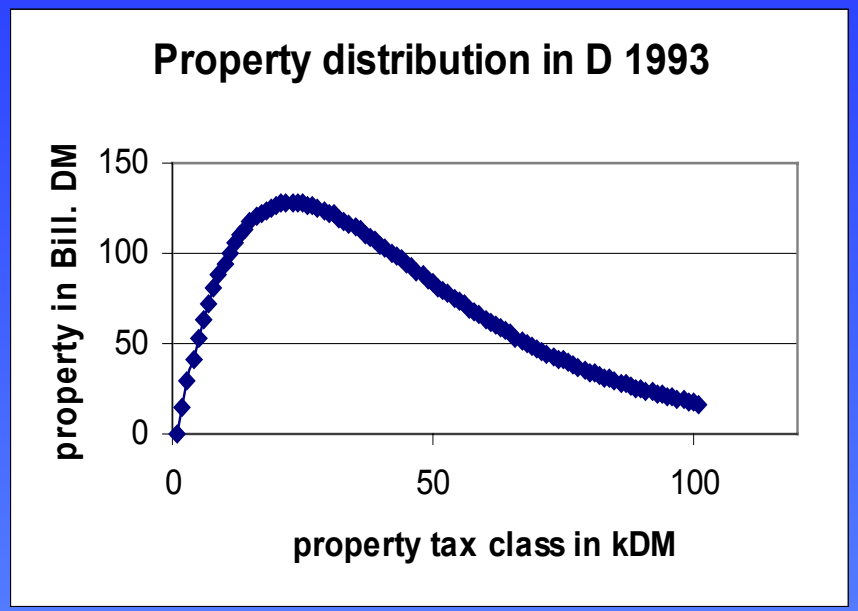
$K = \int k\, N(k)\, dk = A\, T^2$ = 10 Trill.DEM

$T = K / N$ = 278.000 DEM / Hh

Lagrange Principle and Boltzmann Distribution of Wealth

Comparing Boltzmann to DIW wealth data for Germany 1993





$$N(k) = (N_0/T)\ \exp(-k/T)$$

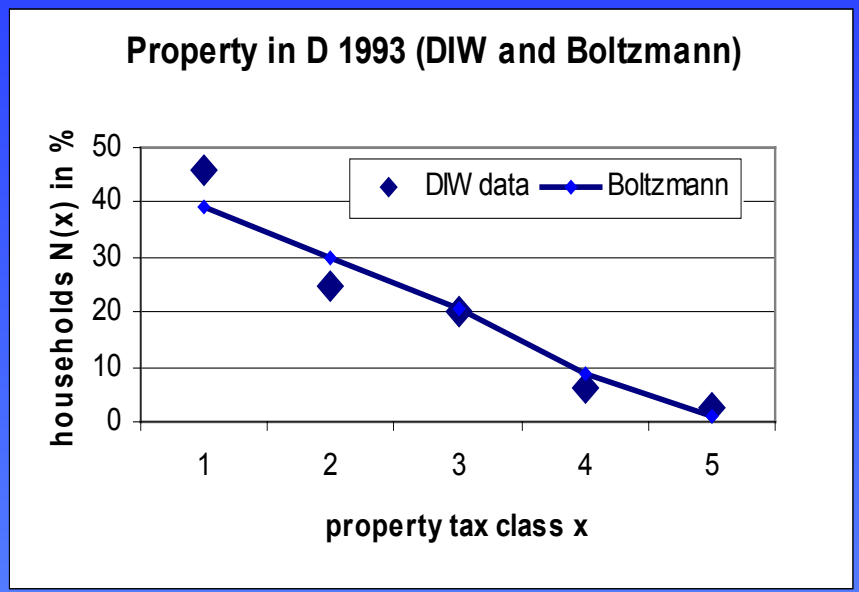
$$\int N(k)\,dk = N_0 = 35{,}6 \text{ Mill. Hh}$$

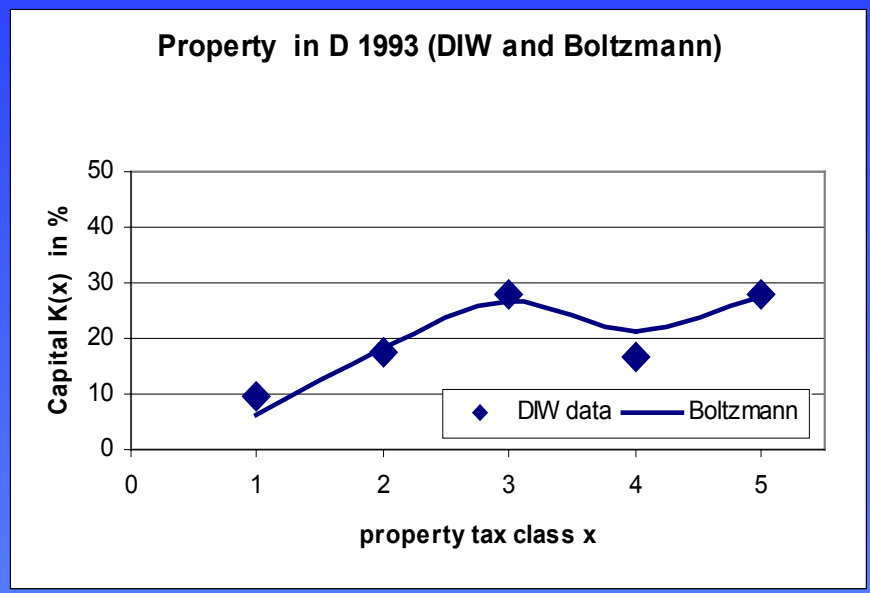
$$K(k) = k\ (N_0/T)\ \exp(-k/T)$$

$$\int K(k)\,dk = N_0\,T = K_0 = 9{,}9 \text{ Trill DM}$$

Lagrange Principle and Boltzmann Distribution of Wealth

Comparing Boltzmann to DIW wealth data for Germany 1993





$$N(k) = (N_0/T)\ \exp(-k/T)$$

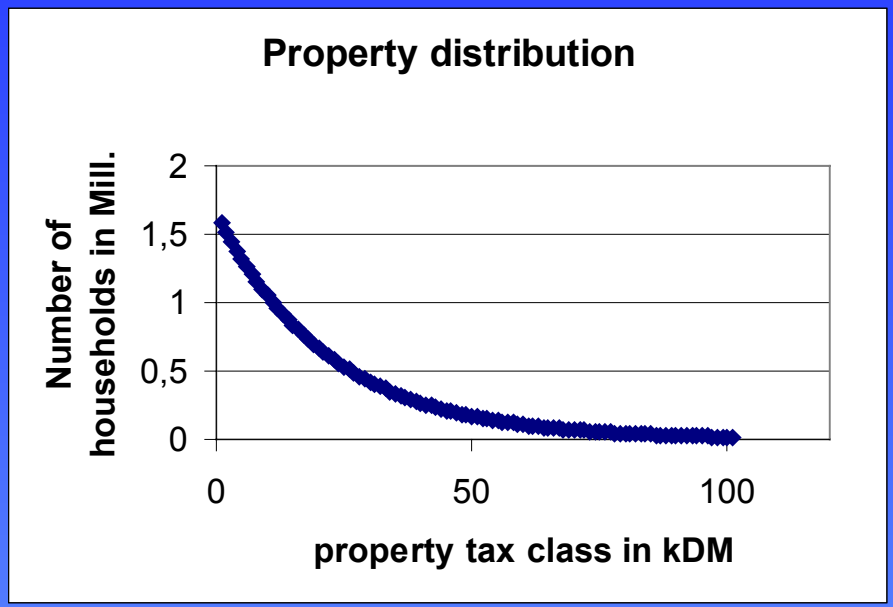
$$\int N(k)\,dk = N_0 = 35{,}6 \text{ Mill. Hh}$$

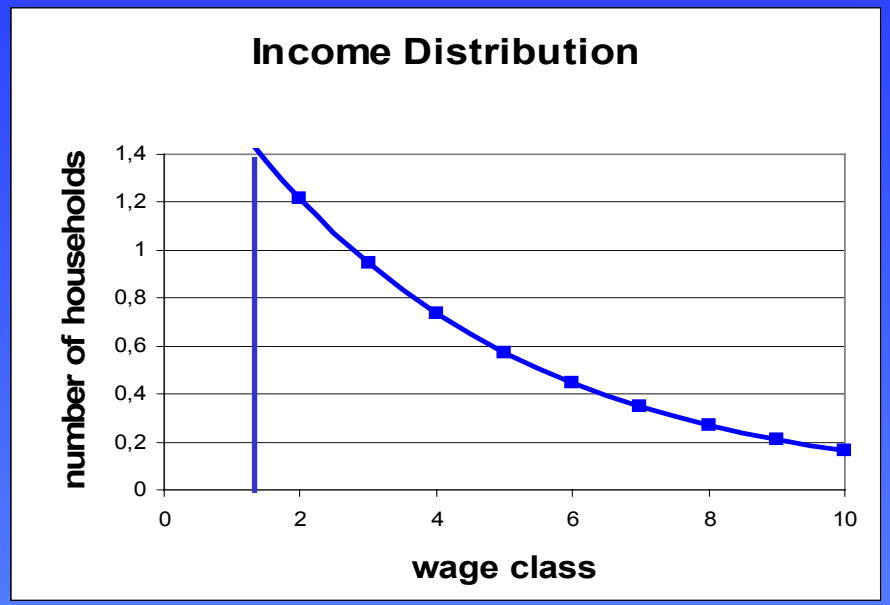
$$K(k) = x\ (N_0/T)\ \exp(-k/T)$$

$$\int K(k)\,dk = N_0 T = K_0 = 9{,}9 \text{ Trill DM}$$

Lagrange Principle and Boltzmann Distribution of Wealth

Comparing Wealth to Income





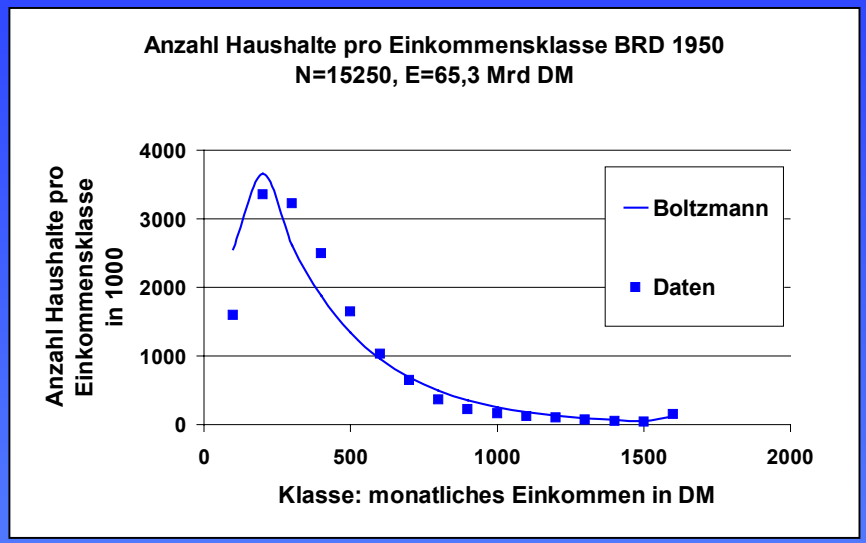
Distribution of wealth

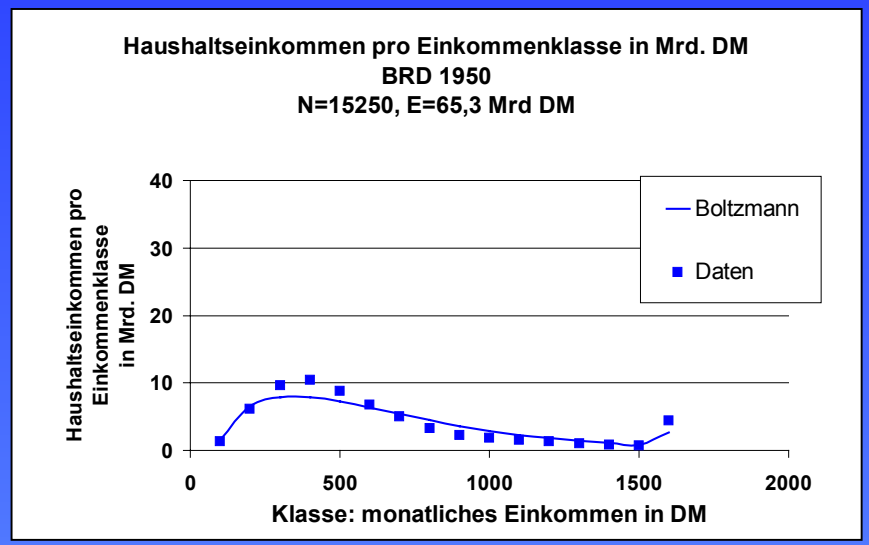
Distribution of income

Lagrange Principle and Boltzmann Distribution of Wealth

1950	Data	Boltzmann
N (Number of households in 1000)	**15250**	**15250**
K (Total income)	**65,3 Mrd DM**	**74,6 Mrd DM**

G. Göseke, Verteilung und Schichtung der Einkommen privater Haushalte in der BRD 1950 - 1975





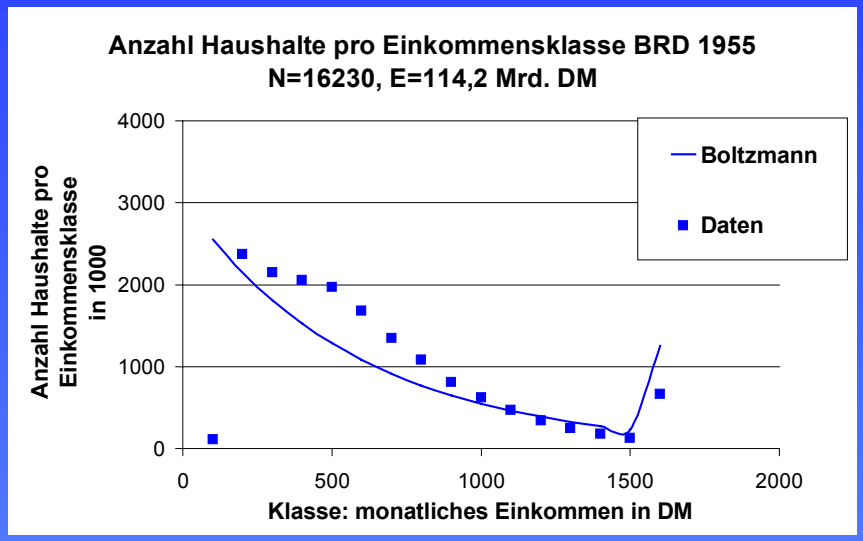
Number of households per income class

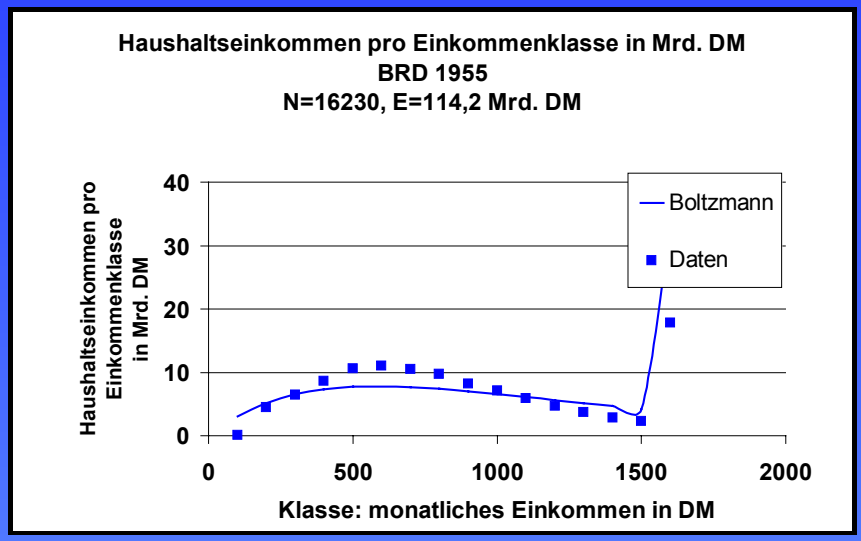
Income per income class

Lagrange Principle and Boltzmann Distribution of Wealth

1955	Data	Boltzmann
N (Number of households in 1000)	**16230**	**16230**
K (Total income)	**114,2 Mrd DM**	**124 Mrd DM**

G. Göseke, Verteilung und Schichtung der Einkommen privater Haushalte in der BRD 1950 - 1975





Number of households per income class

Income per income class

Lagrange Principle and Boltzmann Distribution of Wealth

1960	Data	Boltzmann
N (Number of households in 1000)	**18905**	**19470**
K (Total income)	**190 Mrd DM**	**198 Mrd DM**

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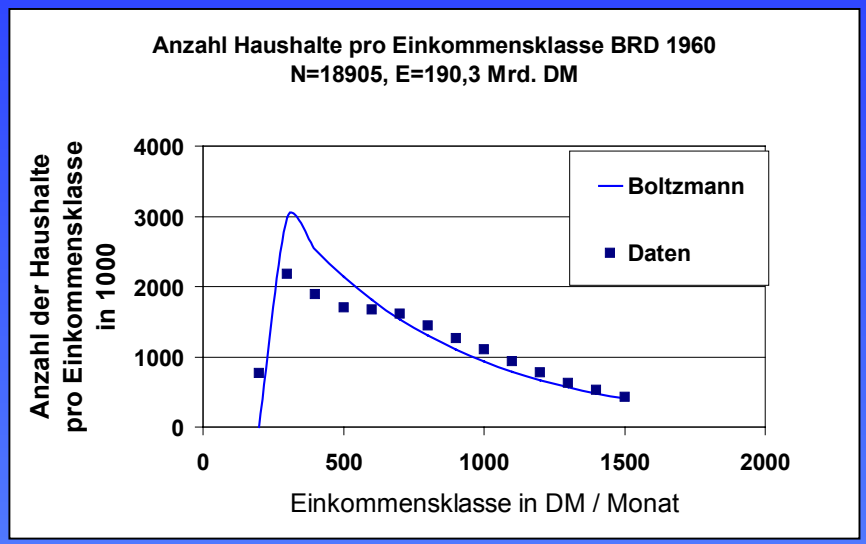
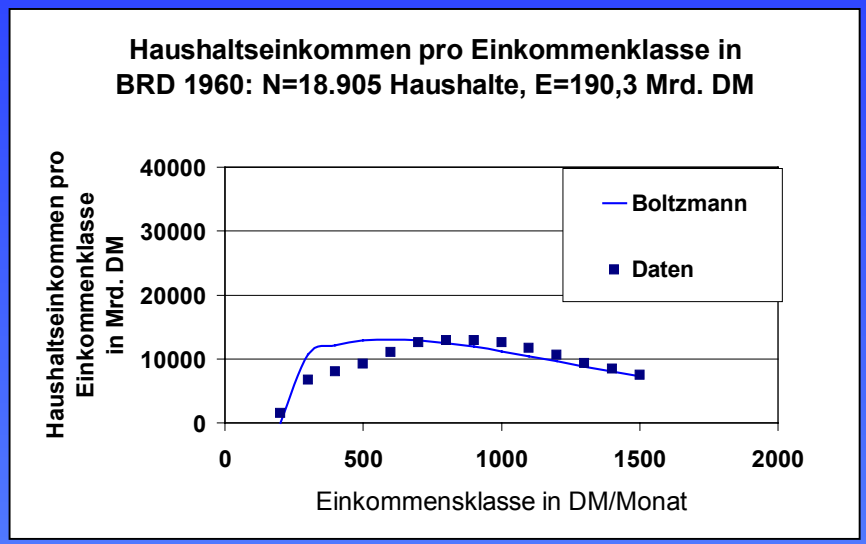




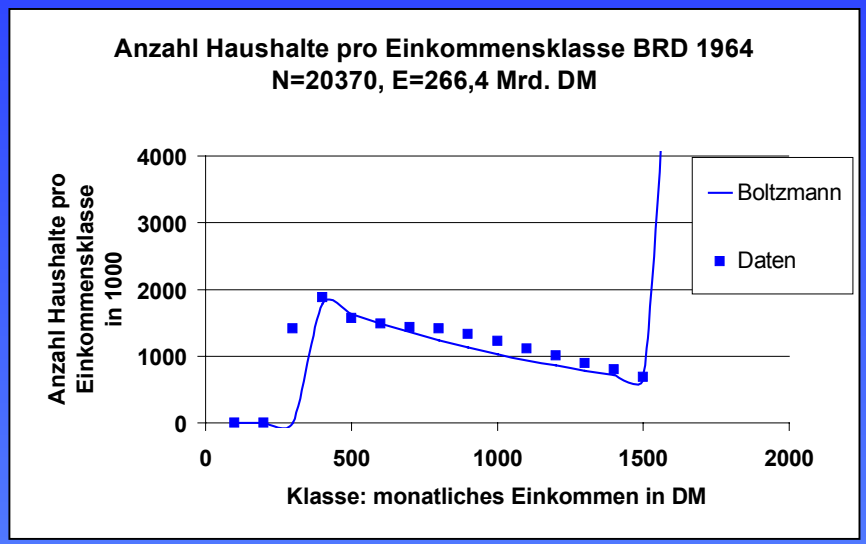
Number of households per income class

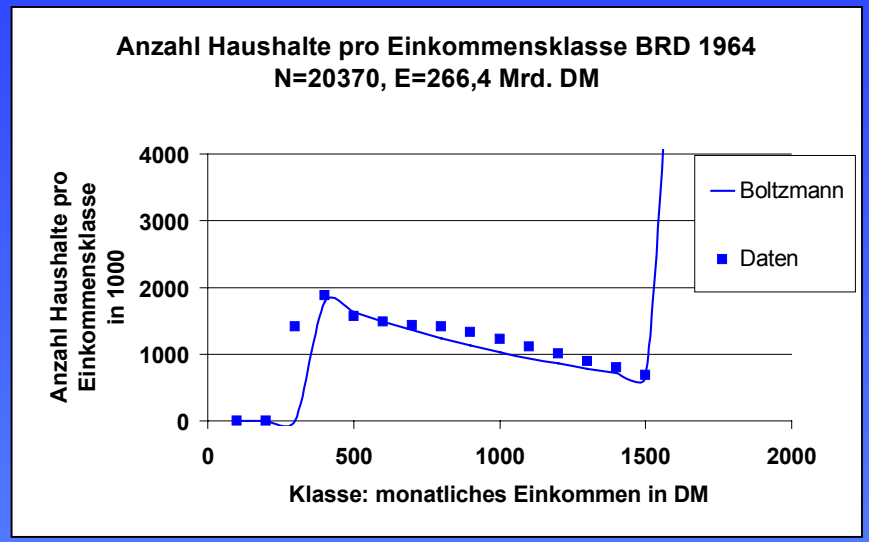
Income per income class

Lagrange Principle and Boltzmann Distribution of Wealth

1964	Data	Boltzmann
N (Number of households in 1000)	**20370**	**20369**
K (Total income)	**266,4 Mrd DM**	**351,8 Mrd DM**

G. Göseke, Verteilung und Schichtung der Einkommen privater Haushalte in der BRD 1950 – 1975





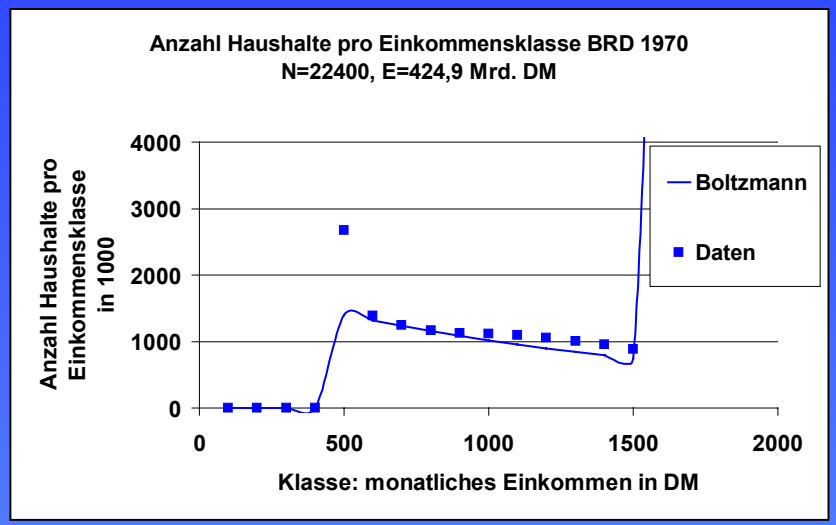
Number of households per income class

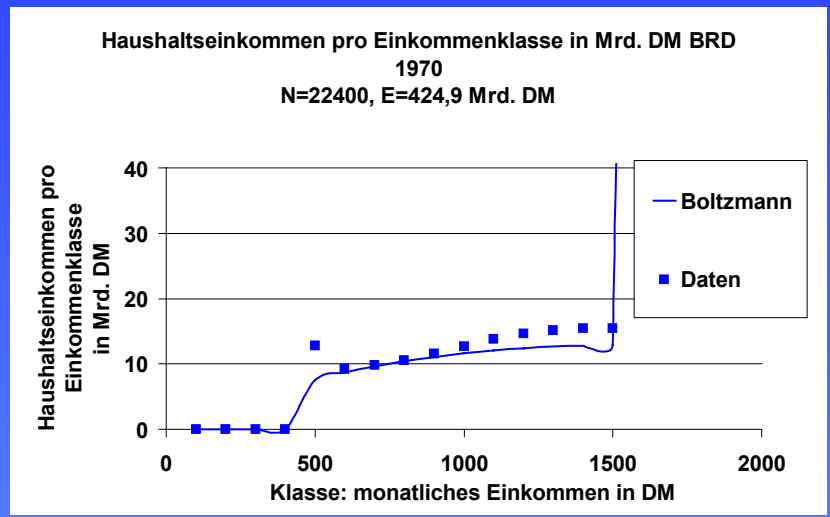
Income per income class

Lagrange Principle and Boltzmann Distribution of Wealth

1970	Data	Boltzmann
N (Number of households in 1000)	**22400**	**22367**
K (Total income)	**424,9 Mrd DM**	**567,8 Mrd DM**

G. Göseke, Verteilung und Schichtung der Einkommen privater Haushalte in der BRD 1950 - 1975





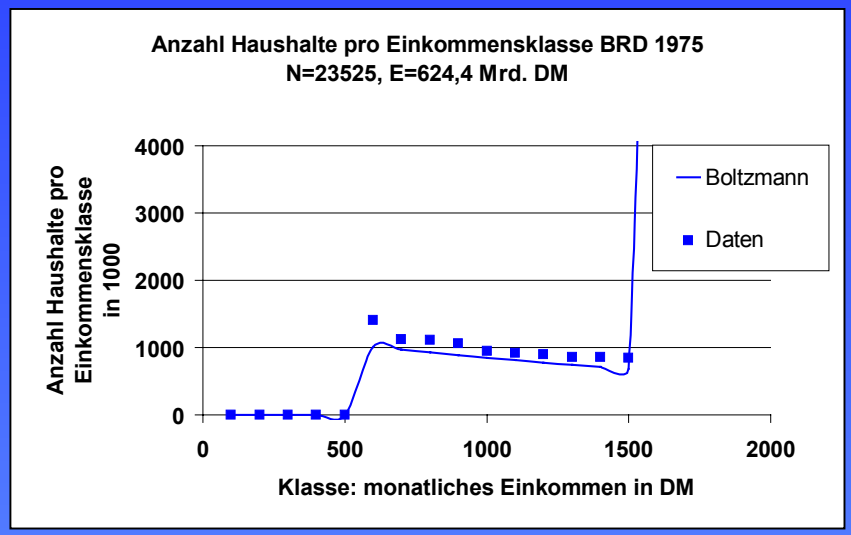
Number of households per income class

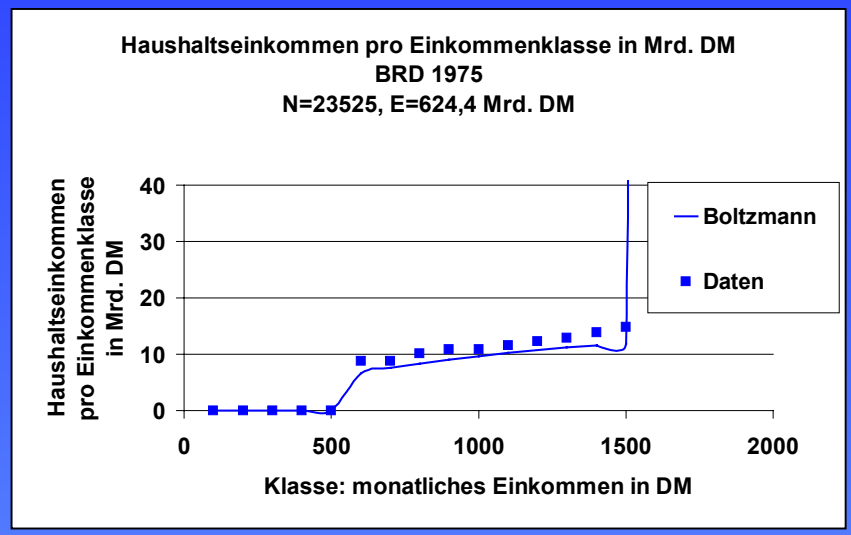
Income per income class

Lagrange Principle and Boltzmann Distribution of Wealth

1975	Data	Boltzmann
N (Number of households in 1000)	**23525**	**23270**
K (Total income)	**624,4 Mrd DM**	**757,6 Mrd DM**

G. Göseke, Verteilung und Schichtung der Einkommen privater Haushalte in der BRD 1950 - 1975

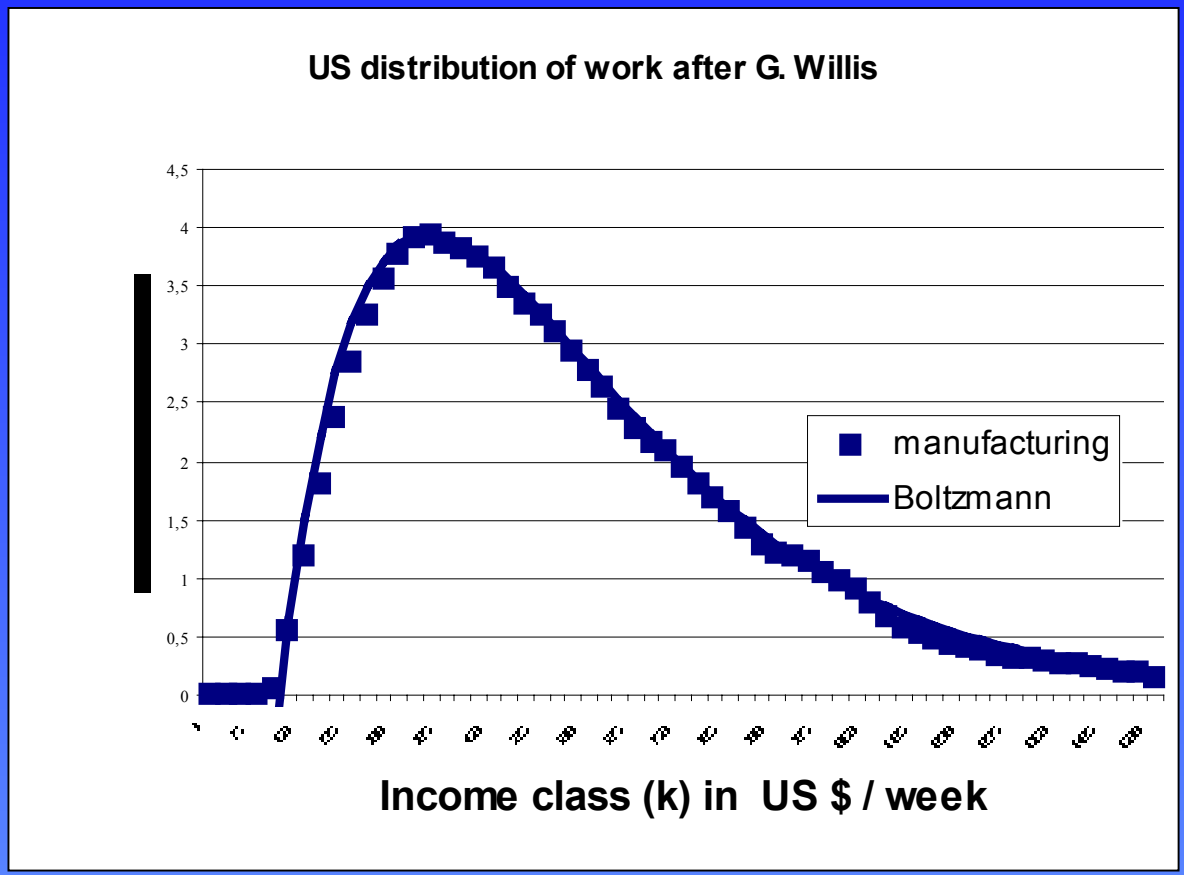




Number of households per income class

Income per income class

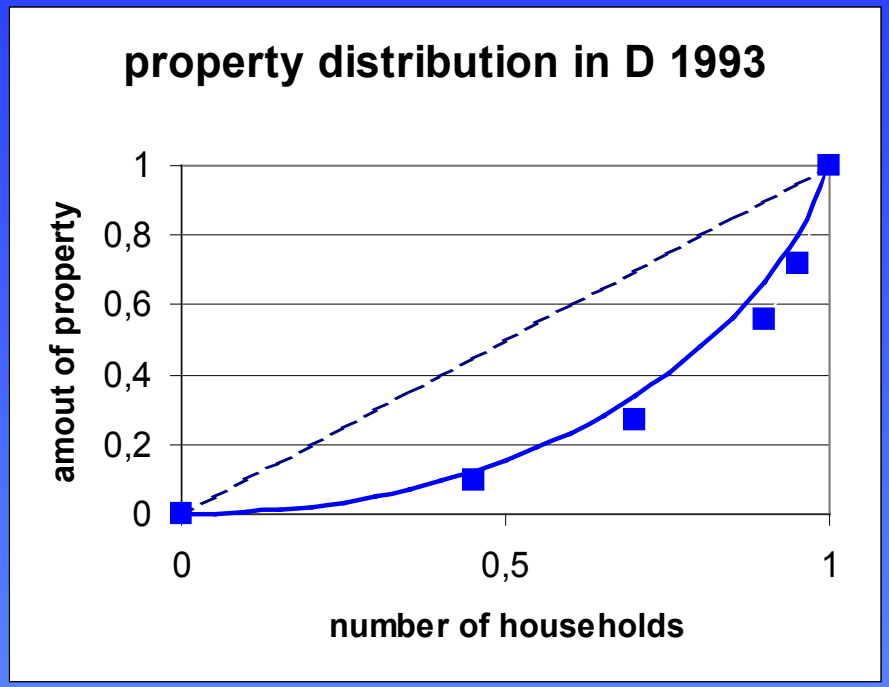
Lagrange Principle and Boltzmann Distribution of Wealth



US income 1995 Distribution of workers N(k)=g(k) exp(-k/T)

Lagrange Principle and Boltzmann Distribution of Wealth

Lorenz distribution of property data for Germany 1993 (DIW)

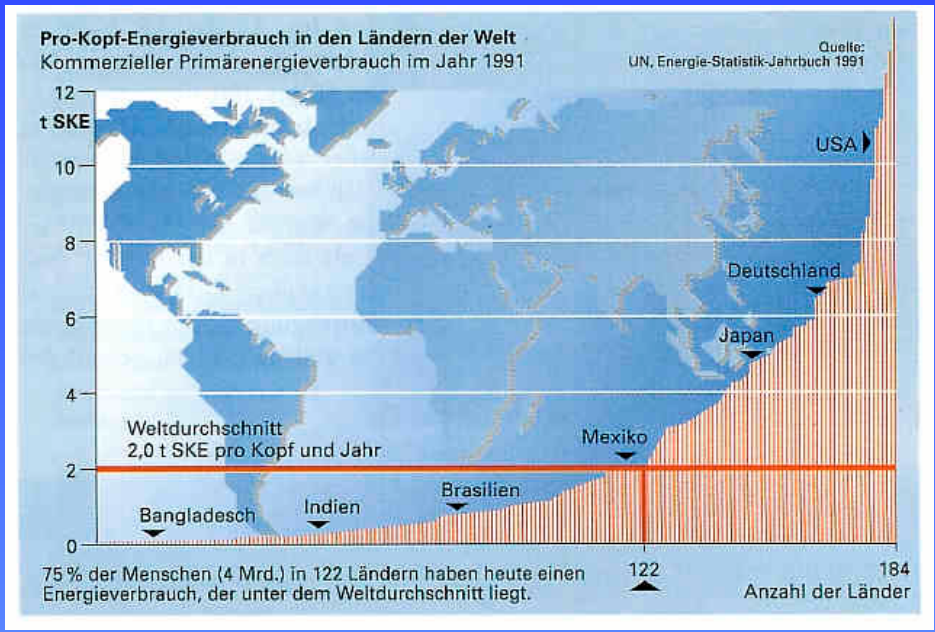


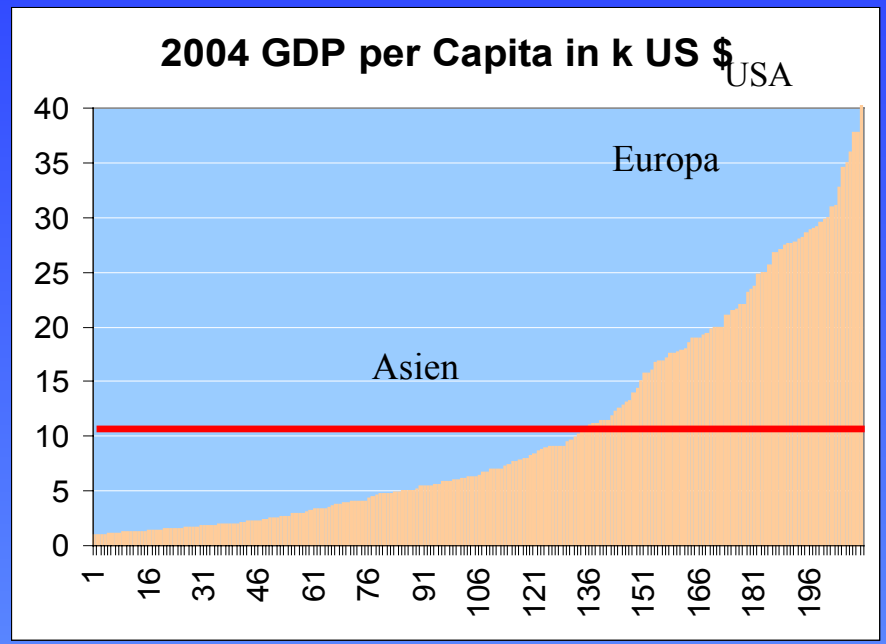
The distribution of wealth is a Boltzmann distribution up to 90 % of households.
At very high incomes an additional function f(x) seems to indicate a two level wealth distribution.
This may be explained by a Carnot production process.

$$y_{\text{Boltzmann}} = x + (1-x)\ln(1-x) + f(x)$$

Lagrange Principle and Boltzmann Distribution of Wealth

Lorenz distribution of world wealth



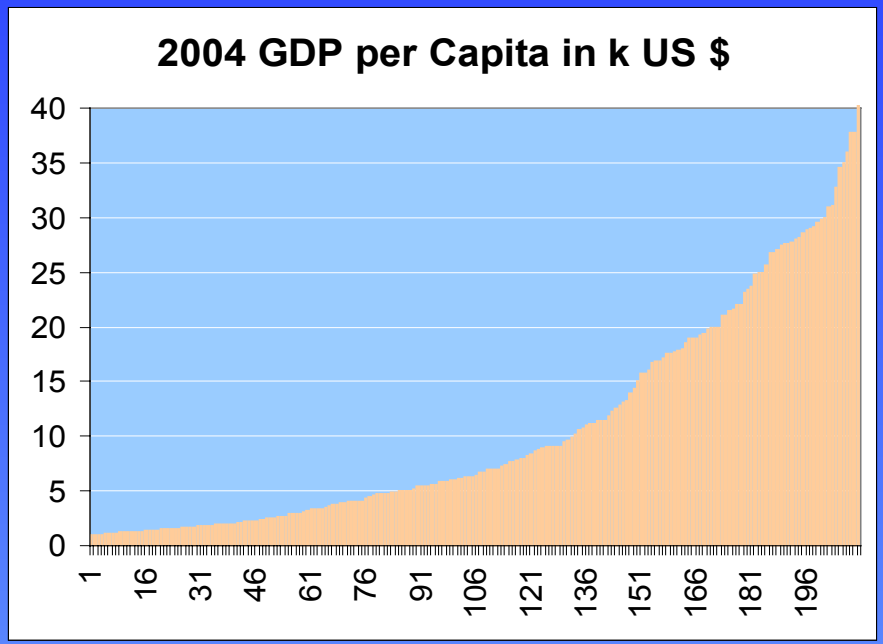


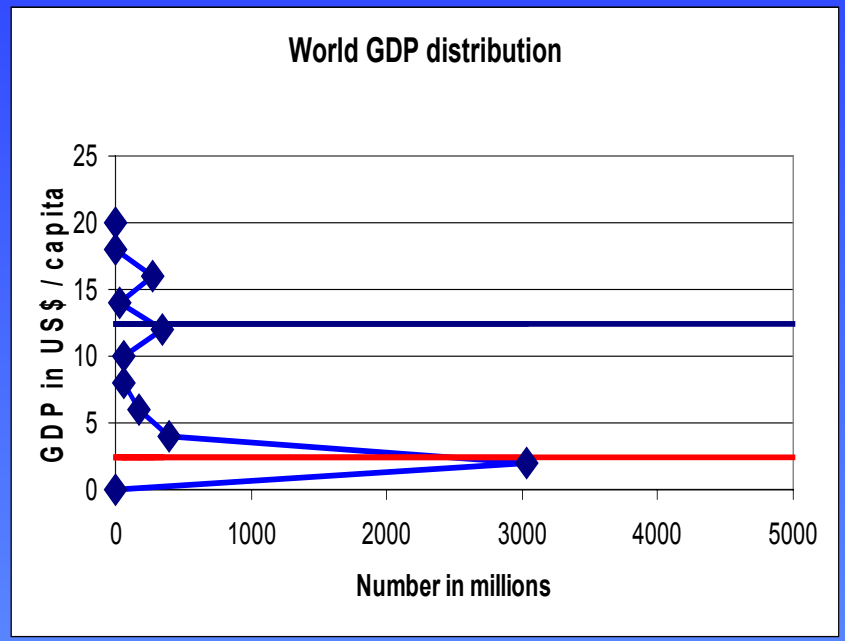
World energy consumption / capita

World GDP / capita

Lagrange Principle and Boltzmann Distribution of Wealth

Distribution of world wealth





A two level distribution of world GDP per capita