How the rich get richer

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Thanks to
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J M Luck (SPhT, Saclay)

Plan of talk

- Model ingredients - n interacting traders via bank
- A tale of two traders
- Infinitely many traders - in a soup! the mean-field limit
- Infinitely many traders - the emergence of monopolists in local neighbourhoods

A model of n interacting traders - basic ingredients

- n traders interact via bank
- Wealth $m_i(t)$ for $i = 1, \ldots, n$ of each trader evolves as:
$$\frac{\mathrm{d}m_i}{\mathrm{d}t} = \left(\frac{\alpha}{t} - \frac{1}{t^{1/2}} \sum_j g_{ij} \frac{\mathrm{d}m_j}{\mathrm{d}t}\right) m_i - \frac{1}{m_i}$$
- Gain term = net gain from bank + net gain from trader transactions
- Loss term $\boxed{\triangleright}$ the less you have, the more you lose!

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- Net gain from bank:
  ▷ Interest on capital at $\alpha > 1/2$ which is taxed so that it is depleted as $1/t$.
- Net gain from trader transactions:
  ▷ Gross amount g_{ij}x(total loss of other traders)
  ▷ This is also taxed, so net amount depleted as $1/t^{1/2}$.

Some notations

- use $s = \ln \frac{t}{t_0}$ (to renormalise away the effect of initial time t_0)
- renormalised wealth $x_i = \frac{m_i}{t^{1/2}}$ and renormalised square wealth $y_i = x_i^2 = \frac{m_i^2}{t}$.

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- A trader who is richer than threshold
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- A trader who is poorer than threshold goes bankrupt and dies...

A tale of two (actually all $n \geq 2$) traders

The socialist case - two equally poor traders

- Common wealth $x(s)$ maintained by symmetry forever:
  $x' = \frac{(2\alpha - 1)x^2 - 2 - gx^3}{2x(1+gx)}$
- Critical value of trading strength g_c such that:
  ▷ for $g > g_c$ (heavy trading, insufficient saving), both traders go quickly bankrupt

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Moral - Only those who are BORN rich stand a chance...

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- When, additionally, $\bar{g}$ is small, ▷ trading is weak,we observe a *glassy* dynamics with two-step relaxation!

Glassy dynamics: a) two-step relaxation

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- **The richest trader 'consumes' all others...** ...and may live to tell the tale... ..ONLY if he was born richer than $y_\star$!
- Two well-separated time scales of fast and slow dynamics ▷ glassy dynamics!

Glassy dynamics: b) universality

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In fact, universality much stronger: only the **tail exponent** of the distribution determines most features.

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- Traders only interact with their $z = 2D$ nearest neighbours on a D-dimensional lattice, with interaction $g \ll 1$.
- Two well-separated stages in dynamics (as before):
  ▷ Fast individual dynamics in Stage I, where traders evolve in isolation (as before)
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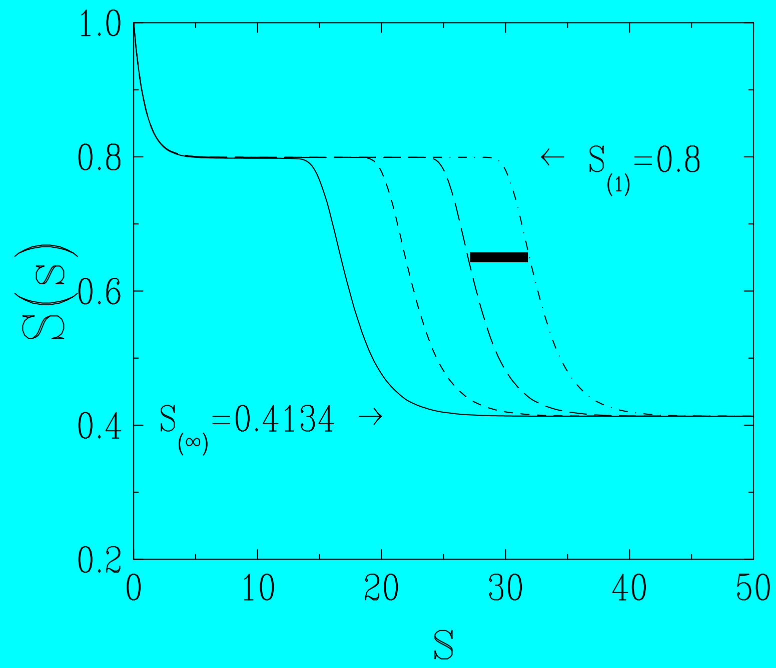
- **however...**
- Effect of neighbourhoods comes into play in slow collective dynamics of Stage II:
  ⊳ $S(s)$ decays from $S_{(1)}$ to non-trivial limiting value $S_{(\infty)}$!!
  ⊳ a *finite* number of traders survive!

Moral: Every 'para' creates a 'mastan'!

Two-step relaxation and ageing

Survival probability versus time for 3 values of g



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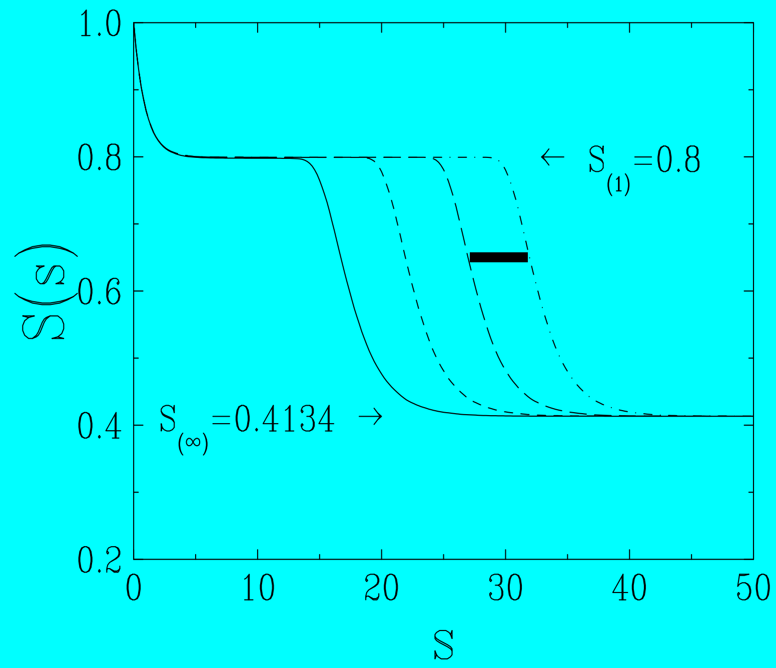

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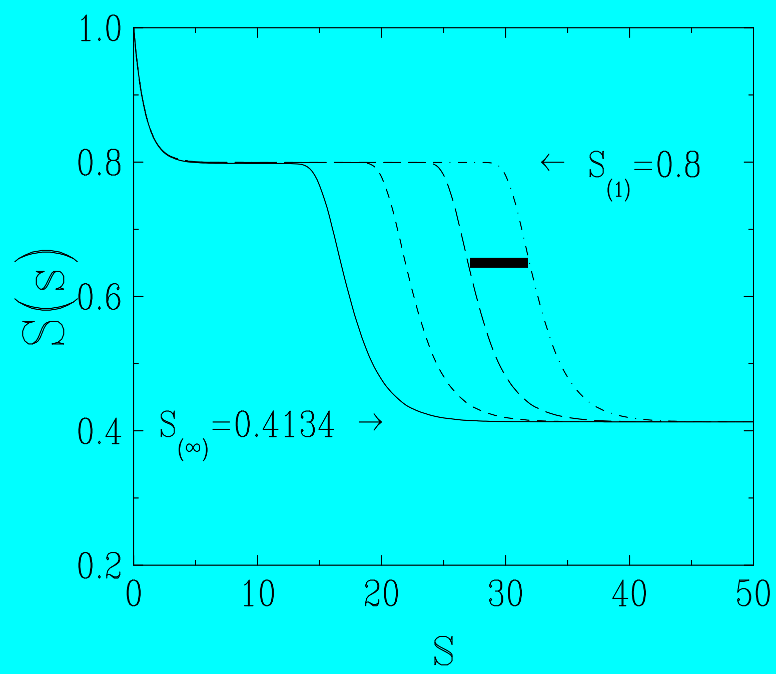

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- The weaker the trading strength, the longer it takes for monopolists to emerge - ageing!

Survivors at $S_{(\infty)}$ = monopolists

- After Stage II, system in non-trivial *attractor* - each trader has a *monopoly*, and keeps getting richer forever !
- Such attractors are *metastable states*, valleys in existing random energy landscape!
- Particular metastable state chosen is that which is most easily be reached...
- Number $\mathcal{N}$ of such states grows exponentially with number of sites N: $\mathcal{N} \sim \exp(N\Sigma)$
  $\boxed{\triangleright}$ Σ is the *complexity*!
- $S_{(\infty)}$ is density of a typical attractor - fraction of initial traders which survive forever!

The fate of a marketplace: patterns of monopolists

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Moral: **Every mastan needs his own space..**

Pattern formation

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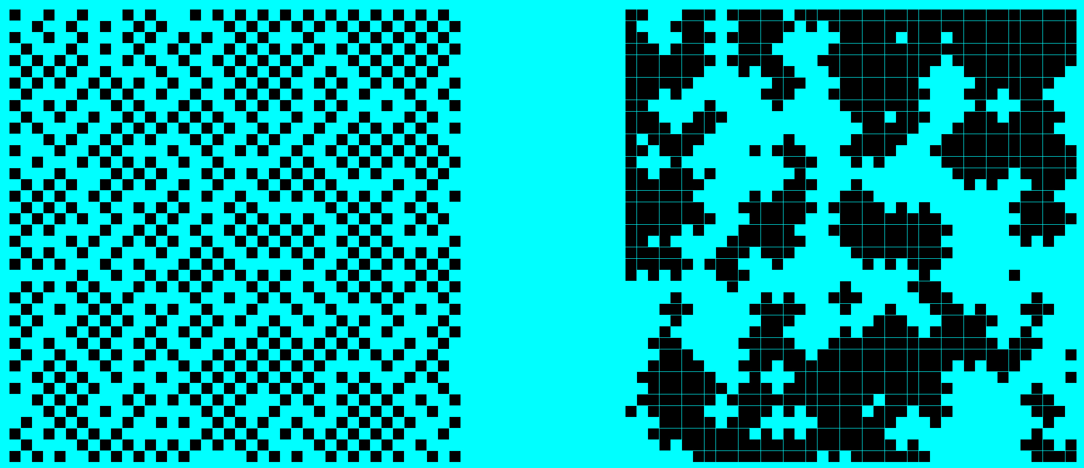
- can define survival index (according to occupation) or checkerboard index (according to parity)



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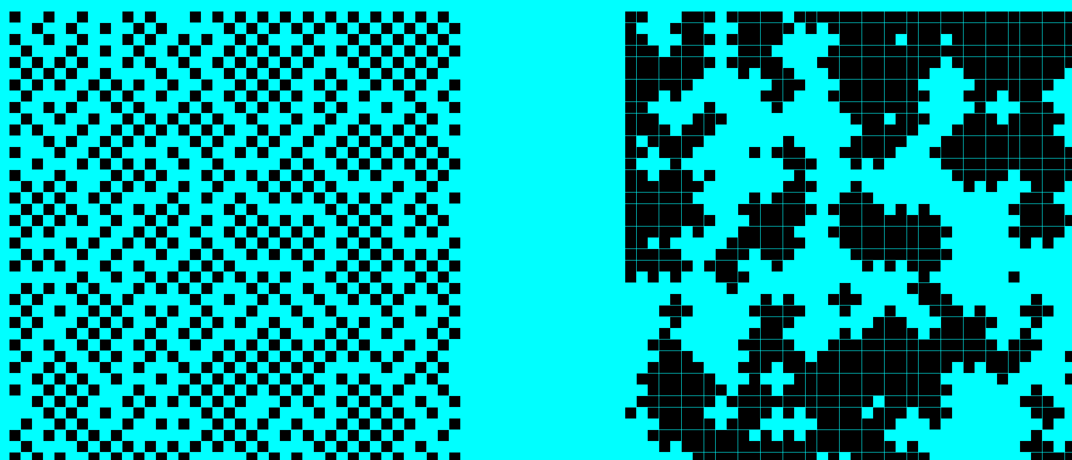

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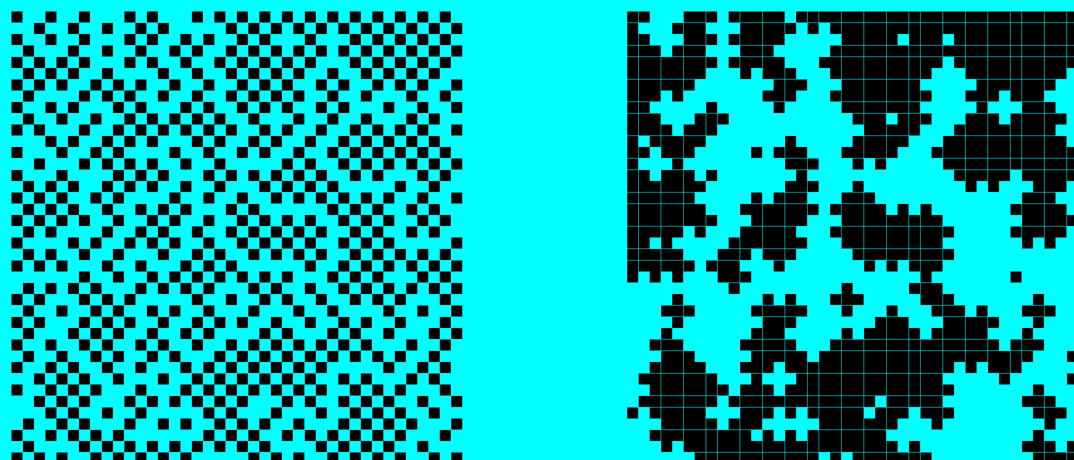

- Pattern above (for $S_{(\infty)} = 0.371$) has local checkerboard structure with frozen-in defects between patterns of different parities
- Random structure entirely inherited from initial wealth distribution
- Nearest neighbours of monopolists are paupers - but next-nearest ones likely to be monopolists!

  ▷ confirmed by correlation functions

Moral: Those who are born rich survive to get richer, while the poor eventually disappear

'Some are born to sweet delight
Some are born to endless night'

William Blake