Nuclear Astrophysics
Lecture 1

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TRIUMF

Education is what remains after you have forgotten everything you have learned.

$$-i\varepsilon$$

Nuclear Astrophysics

The luminous structure of the Universe is largely given by the properties of the nuclei inhabiting it. These provide the energy source and thus the structure of stars and produce this way new nuclei and elements.

Nuclear Physics explains the working of nuclei. Many properties of nuclei and nuclear scattering are known from experiments with no particular interest in astrophysical questions.

Nuclear Astrophysics is generally the nuclear physic dedicated to questions arising directly from astrophysical problems.

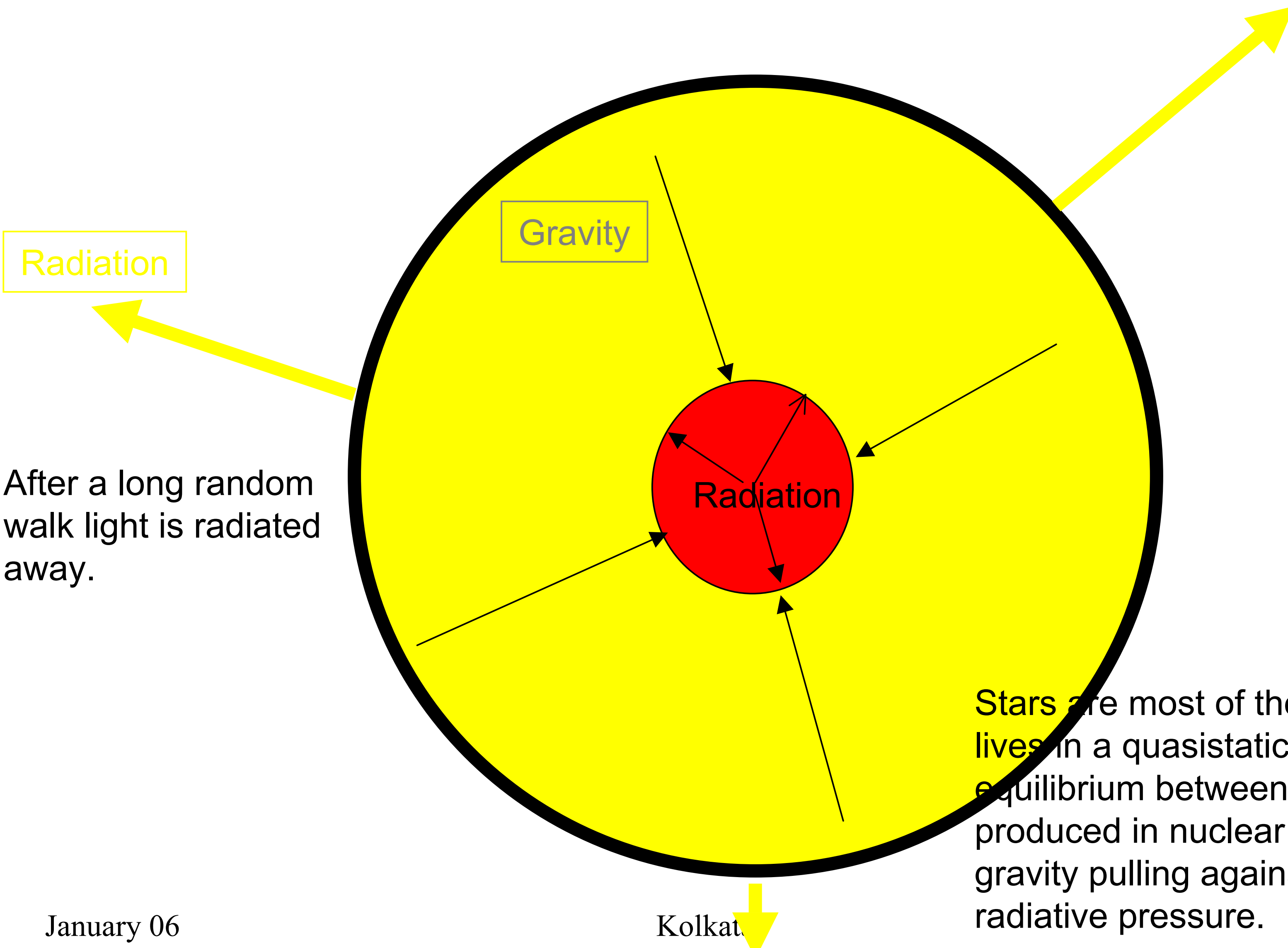
However, the field is rather broad and only a few problems will be discussed here. E.g. there will be no discussion of neutron physics, e.g. capture cross sections, r-process or others.

Overview of lectures

1. A little stellar astronomy
2. A bit more on scattering theory
3. $^{12}C(\alpha,\gamma)^{16}O$, some discussion, new results
4. $^{40}Ca(\alpha,\gamma)^{44}Ti$
5. **$^{7}Be(p,\gamma)^{8}B$** experiment
6. $^{7}Be(p,p)^{7}Be$ and TACTIC
7. Radioactive beam experiments at TRIUMF

1st day

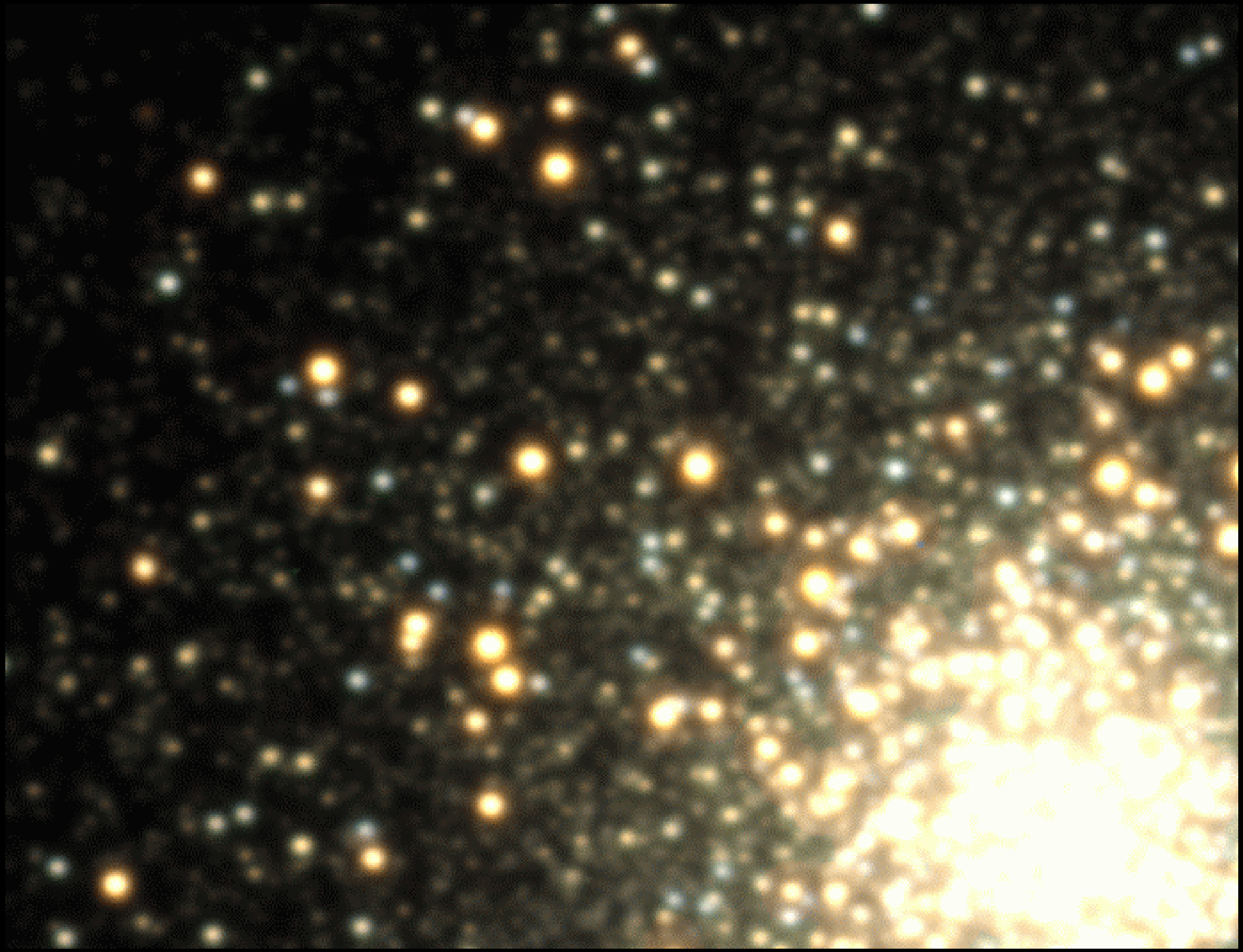
A Star



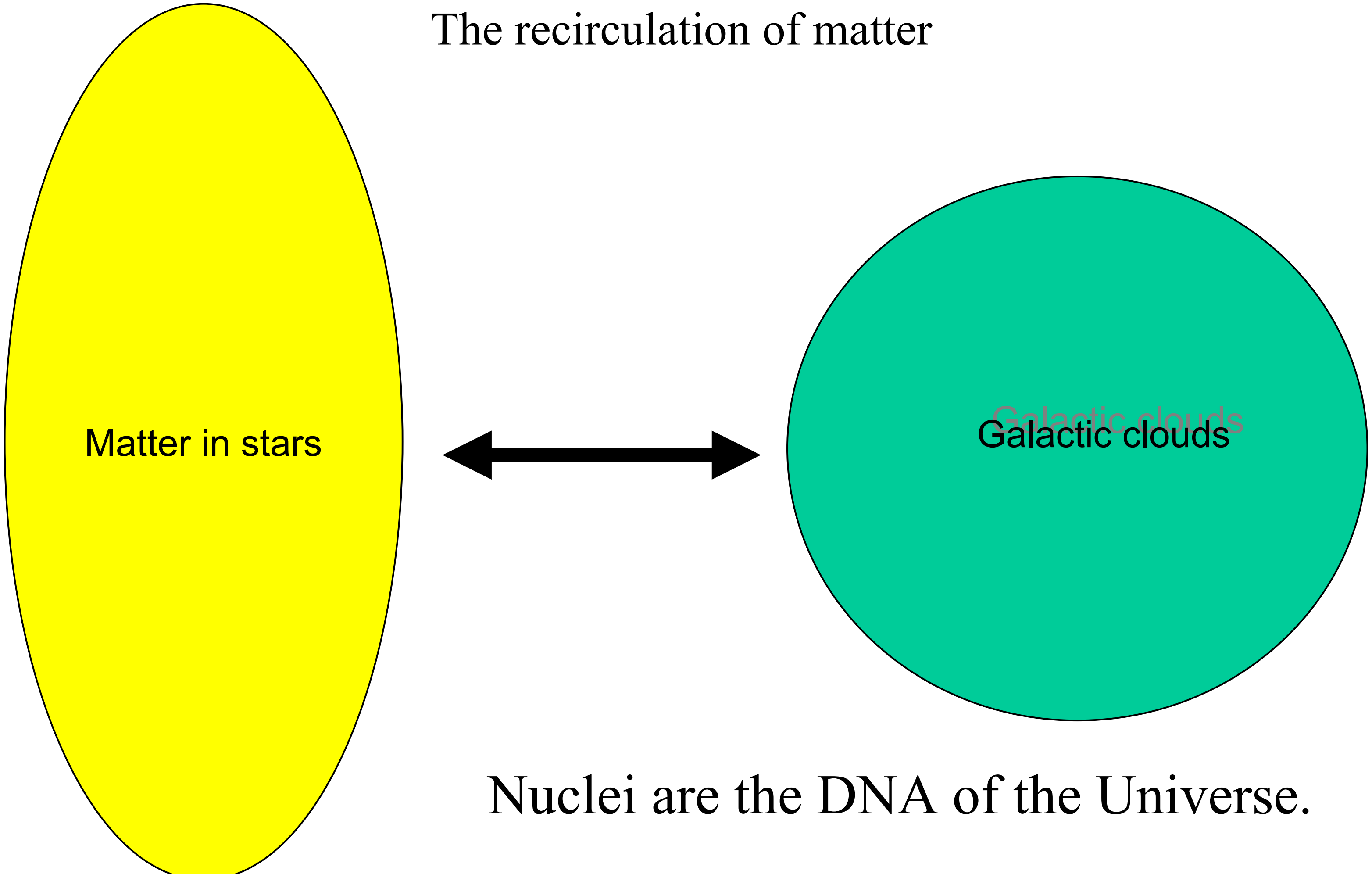
After a long random walk light is radiated away.

Stars are most of their lives in a quasistatic equilibrium between radiation produced in nuclear burning and gravity pulling against the radiative pressure.

Stars twinkle



The recirculation of matter



Nuclei are the DNA of the Universe.

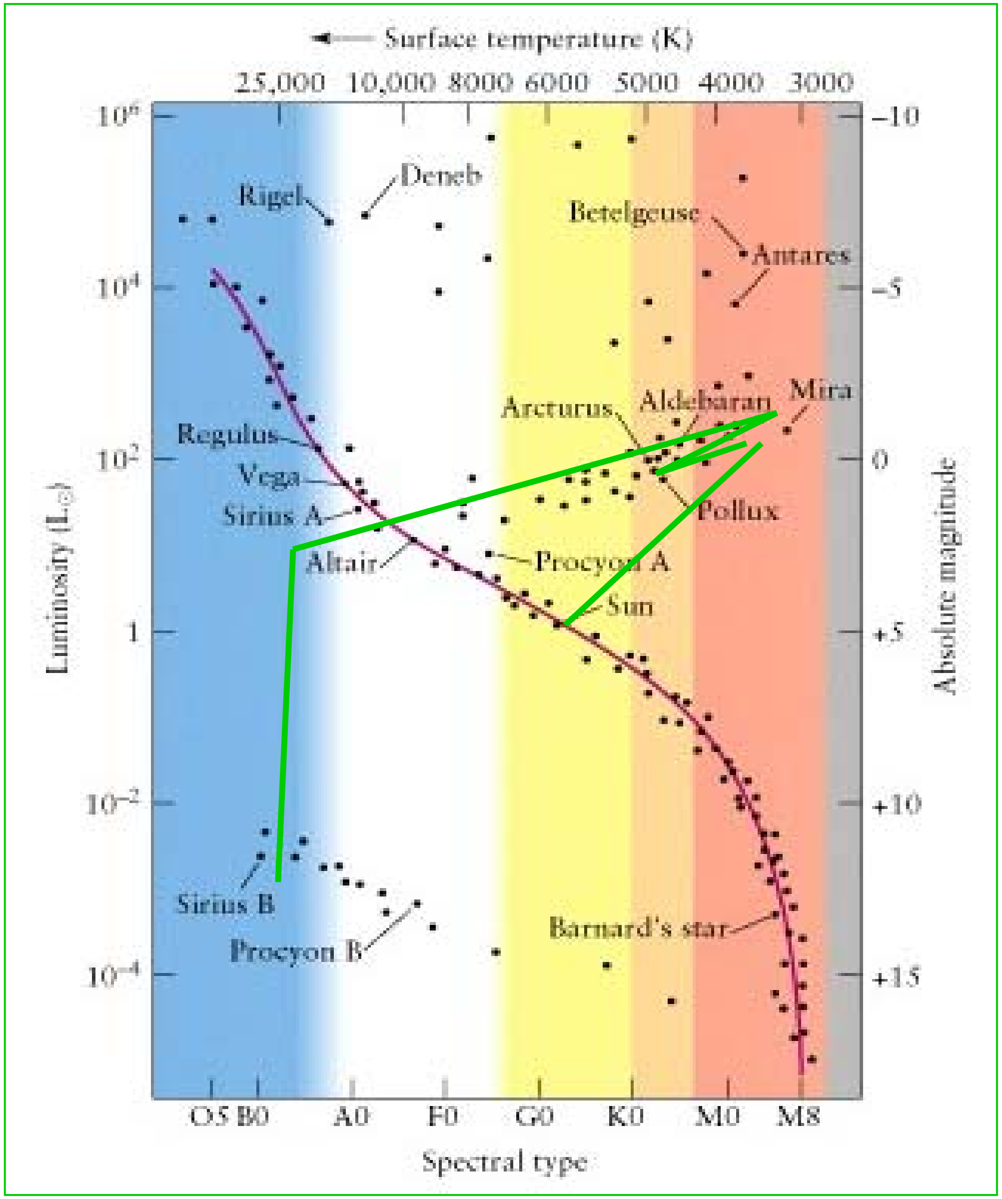
Stan Woosley

Classifying stars: The Hertzsprung-Russell Diagram

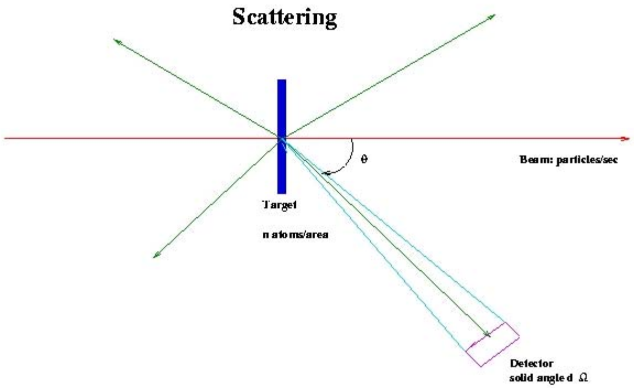
Stars can be classified according to surface temperature and luminosity.

Stellar evolution leads to a path through the HRD.

Stellar evolution is directly linked to microscopic, nuclear physics.

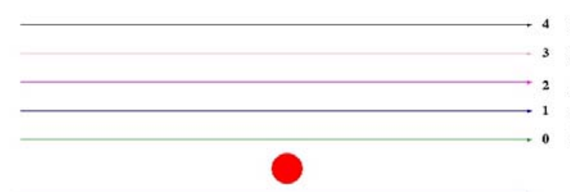


The microscopic world: particle scattering



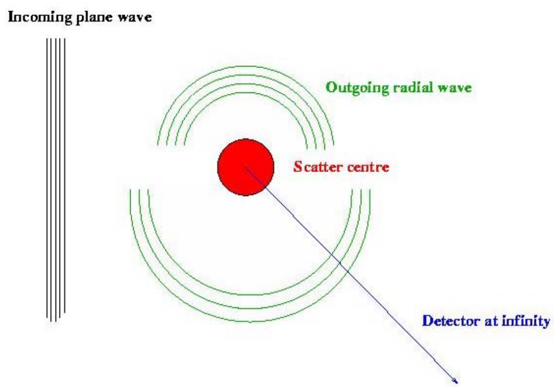
$$Y = I \times n \times \sigma(\theta, \Omega) \times d\Omega$$

$$[\frac{d^2}{dr^2} + (k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2m}{\hbar^2}V(r))]y_\ell(r) = 0$$



Partial wave decomposition

What are Phaseshifts?

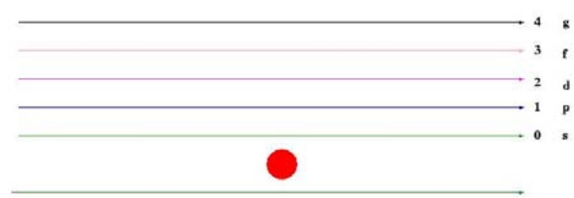


$$\psi(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{z}} + f(\Omega)\frac{e^{ikr}}{r}$$

Radially divergent outgoing wave

$\sigma(\Omega) = |f(\Omega)|^2$ Cross section—square of scattering amplitude

$$f(\theta) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(\cos\theta)$$



Partial wave decomposition

$\updownarrow$

$$f_\ell = \frac{1}{k}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \longleftrightarrow \sigma_{tot} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$

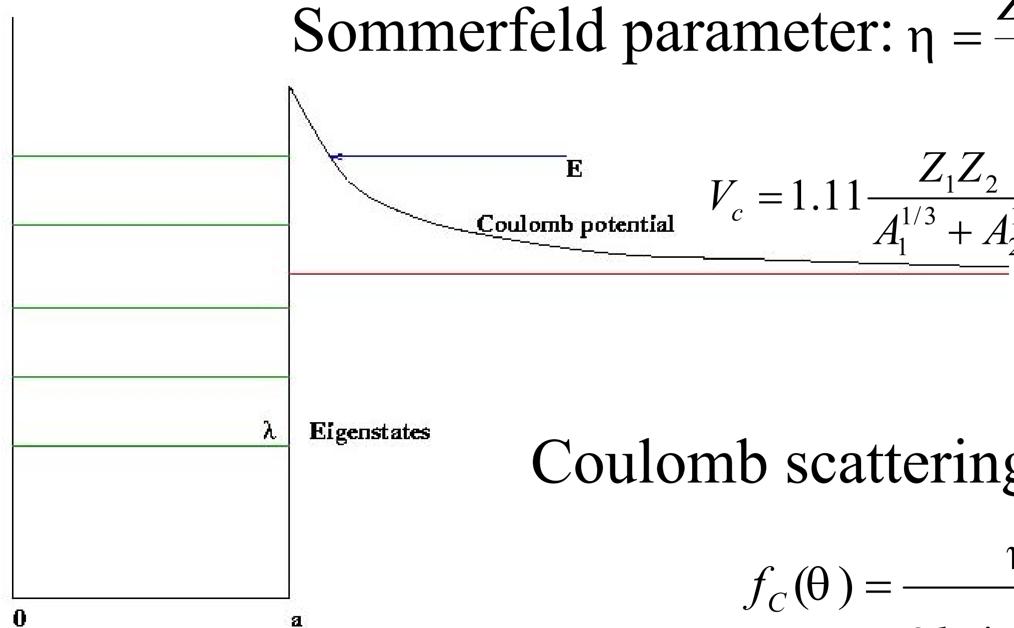
with the phaseshift δ_ℓ

Strongly decreasing potential, no Coulomb forces, no spins.

Coulomb Potential

Infinite range: needs special treatment.

Sommerfeld parameter: $\eta = \dfrac{Z_1 Z_2 e^2}{\hbar v} = 0.157486 Z_1 Z_2 \sqrt{\dfrac{m[\mathrm{u}]}{E[\mathrm{MeV}]}}$



$$V_c = 1.11 \frac{Z_1 Z_2}{A_1^{1/3} + A_2^{1/3}} [\mathrm{MeV}]$$

Coulomb scattering amplitude:

$$f_C(\theta) = \frac{\eta}{2k \sin^2(\frac{1}{2}\theta)} \exp[-i\eta \ln(\sin^2(\frac{1}{2}\theta)) + 2i\varpi_0]$$

More solutions to the Coulomb problem

$$\frac{d\sigma(\theta,E)}{d\Omega} = \frac{1}{k^2}\,\Big| -\frac{\eta}{2}\sin^2\frac{1}{2}\theta\,\exp[-2i\eta\ln\sin(\frac{1}{2}\theta)]$$

$$+\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell\exp(i[2\omega_\ell+\delta_\ell(E)])\sin\delta_\ell(E)\Big|^2$$

Phase shift analysis

Spinless particles, with $\omega_\ell = \sum_{m=1}^{\ell}\arctan(\frac{\eta}{m}),\quad \ell>0,\quad \omega_0=0$

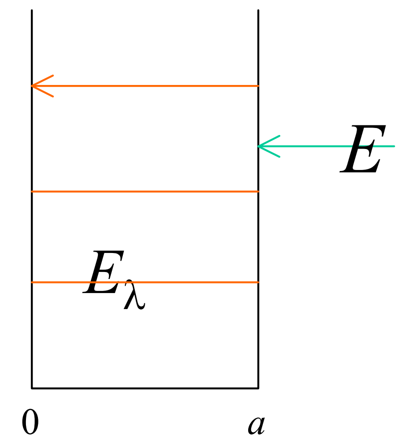
Radial Schrödinger equation for Coulomb problem:

$$y_\ell'' + [k^2 - \frac{2\eta k}{r} - \frac{\ell(\ell+1)}{r^2}]y_\ell = 0$$

Two solutions: $F_\ell(\rho,\eta),..,G_\ell(\rho,\eta)$ regular and irregular Coulomb functions, $\rho=kr$

Derivation of the R-function

Spinless case, one channel, states in a square potential;
then set of eigenstates fulfilling the Schrödinger equation:



$$-\frac{\hbar^2}{2m}\frac{d^2\chi_\lambda}{dr^2}+V(r)\chi_\lambda=E_\lambda\chi_\lambda$$

These can obey a boundary condition at any matching radius:

$$\frac{d\chi_\lambda}{dr}_{|a}+B\chi_\lambda=0$$

An unbound particle scattering at the potential has to also fulfill the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\phi}{dr^2}+V(r)\phi=E\phi \qquad E>0$$

Derivation of the *R*-function

From completeness and orthonormality follows:

$$\phi = \sum_\lambda A_\lambda \chi_\lambda \qquad A_\lambda = \int_0^a \chi_\lambda \phi dr$$

Multiplying the two Schrödinger equations with the opposite wavefunction and subtraction as well as integration from 0 to *a* gives:

$$\int_0^a (\chi_\lambda \frac{d^2\varphi}{dr^2} - \varphi \frac{d^2\chi_\lambda}{dr^2}) dr + \frac{2m}{\hbar^2}(E - E_\lambda)\int_0^a \phi\chi_\lambda dr = 0$$

Partial integration leads to:

$$(\chi_\lambda \frac{d\phi}{dr} - \phi \frac{d\chi_\lambda}{dr})_{r=a} + \frac{2m}{\hbar^2}(E - E_\lambda) A_\lambda = 0$$

Applying the boundary condition and resolving leads to:

$$\varphi(r) = G(r,a)\{\varphi'(a) + B\varphi(a)\}$$

Derivation of the *R*-function

With the Green's function

$$G(r,a) = \frac{\hbar^2}{2m} \sum_{\lambda} \frac{\chi_\lambda(r)\chi_\lambda(a)}{E_\lambda - E} \qquad R \equiv G(a,a)$$

Defining:

$$\gamma_\lambda^2 = \frac{\hbar^2}{2m} \chi_\lambda^2(a) \Rightarrow \quad R = \sum_{\lambda} \frac{\gamma_\lambda^2}{E_\lambda - E}$$

The logarithmic derivative at *a* is then:

$$\frac{\varphi'(a)}{\phi(a)} = \frac{1 - BR}{R}$$

At *a* the internal wavefunctions are matched to the external one:

$$\frac{\psi'(a)}{\psi(a)} = \frac{\varphi'(a)}{\varphi(a)} = \frac{I' - U_\ell O'}{I - U_\ell O} = \frac{1 - BR}{R}$$

Scattering Matrix

With defining the scattering matrix **U:** $$\psi = I - \mathbf{U} O$$

I incoming wavefunction, *O* outgoing.

In single channel *R*-matrix theory is then: $$U_\ell = \frac{I_\ell}{O_\ell} \frac{1 - L_\ell^* R}{1 - L_\ell R}$$

The rest Coulomb properties:

$$I_\ell = (G_\ell - iF_\ell) e^{i\omega_\ell}, \qquad O_\ell = (G_\ell + iF_\ell) e^{-i\omega_\ell}$$

$$L_\ell = S_\ell + iP_\ell$$ with

$$P_\ell = \frac{\rho}{F_\ell^2 + G_{\ell|a}^2}$$

Penetrability

$$S_\ell = P_\ell (F_\ell F_\ell' + G_\ell G_\ell')$$

Shift function

Generalize the scattering matrix

The scattering matrix can be generalized for particles with spins and reactions with multiple channels, so that

for non elastic reactions is (generalization for elastic possible):

$$\sigma_{\alpha\alpha'} = \frac{\pi}{k_\alpha^2} \sum_{Jll'ss'} g_J \, | U^J_{\alpha sl, \alpha' s' l'} |^2$$

With the following symbols:

α, α' physical channel (p, α, others) ; k-wave number

J total spins of (compound) states; **U** –scattering matrix

s incoming channel spin

l angular momentum

g_J spin statistical factor

i.e., elements of the scattering matrix are square roots of reduced cross sections for individual quantum numbers.

What are Phaseshifts?

Then, more formally, phaseshifts are:

$$U^{J}_{\alpha sl,\alpha(')sl} = e^{2i(\omega_{\alpha l}+\delta^{J}_{\alpha sl})}$$

with $\omega_{\alpha l}$ being the Coulomb phase and

$\delta^{J}_{\alpha sl}$ the nuclear phaseshift, usually a complex number, real part elastic, imaginary part sum of all inelastic channels..

As far as elastic scattering is concerned a Coulomb term and a Coulomb-nuclear interference term have to be added.

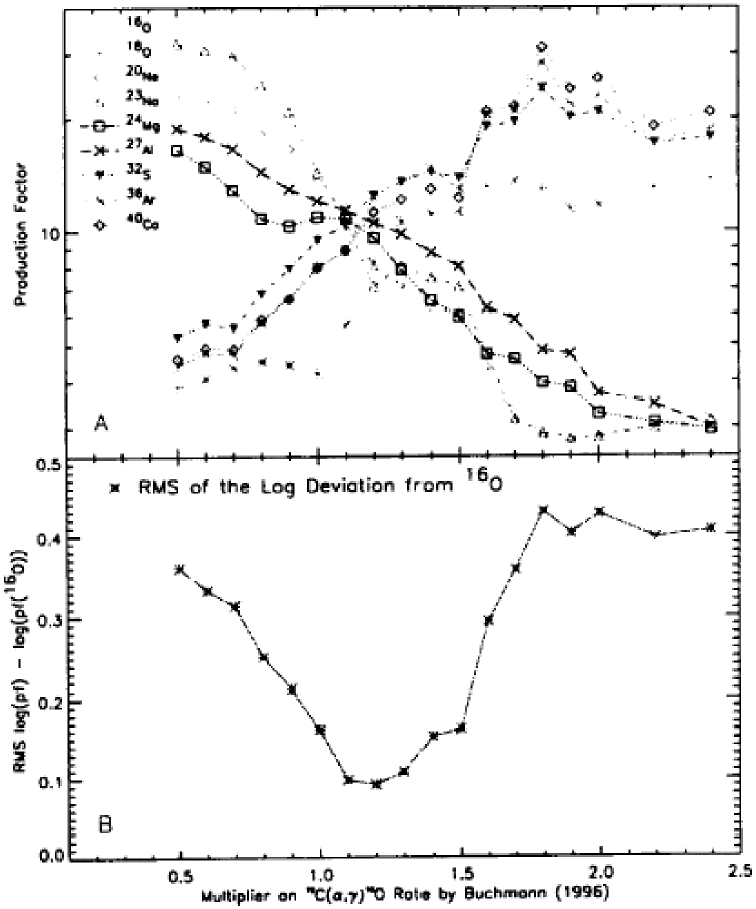
Helium Burning

- When hydrogen burning has stopped in the core it starts to contract slowly till helium ignites. -> “Helium Burning”. The process encompasses only two nuclear reactions as far as energy production is concerned.

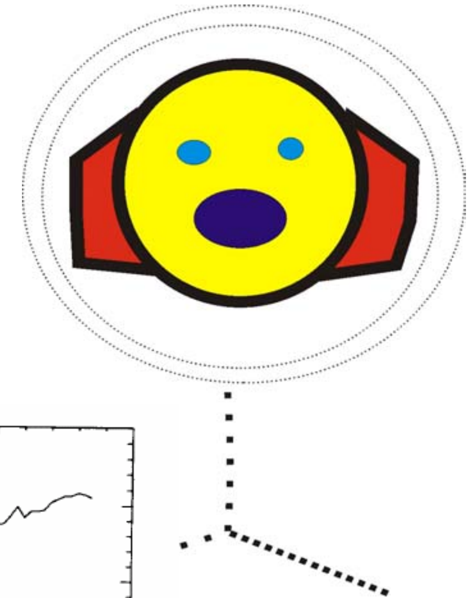
$$3\alpha \rightarrow {}^{12}\mathrm{C}$$

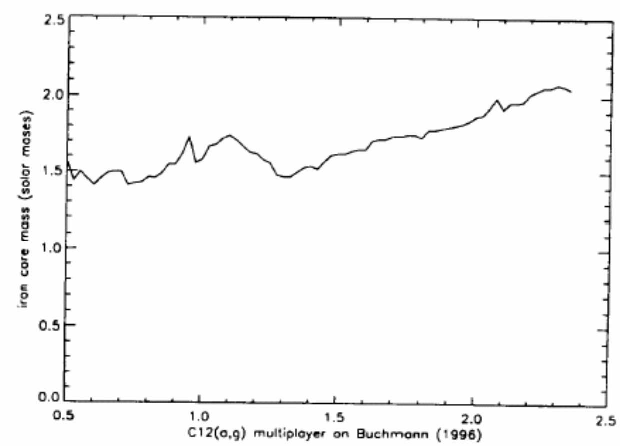
$${}^{12}\mathrm{C}(\alpha,\gamma)^{16}\mathrm{O}$$

Importance of $^{12}C(\alpha,\gamma)^{16}O$



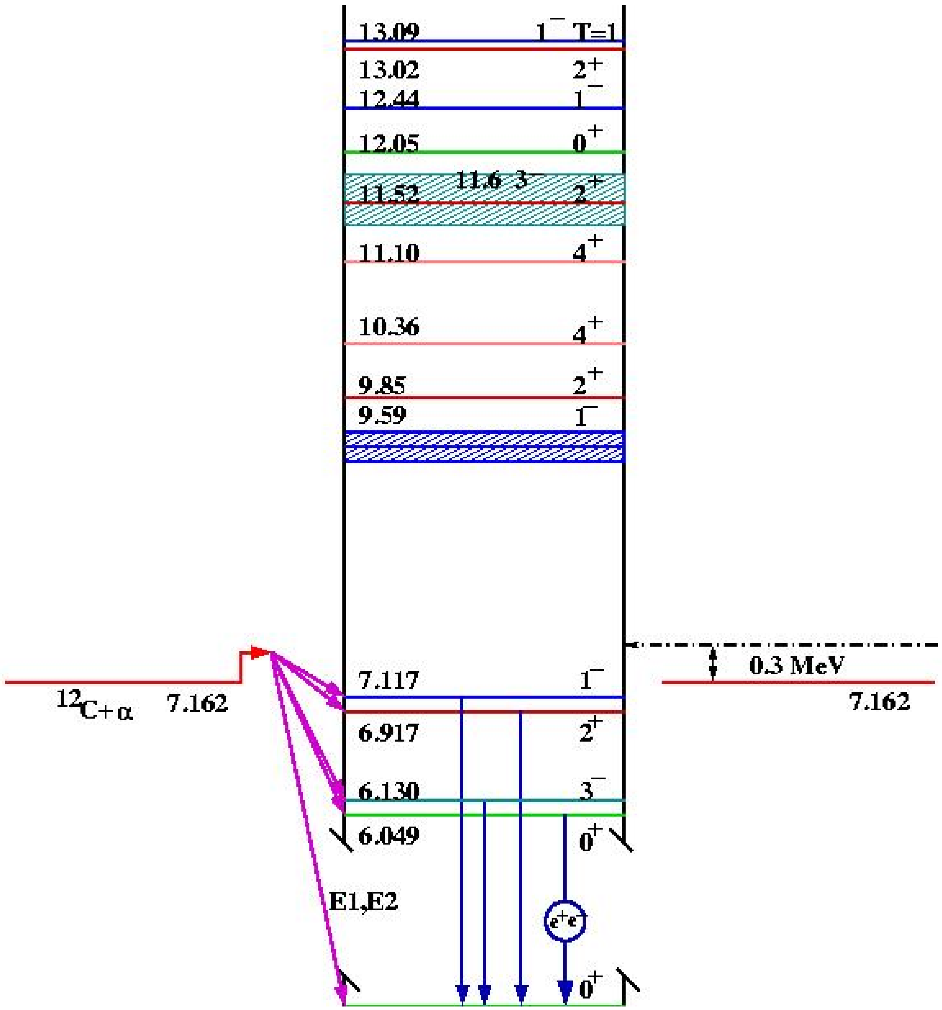
S. Woosley, 2002





SNIa pre-explosion development

Relevant ^{16}O state structure



300 keV corresponds to quiescent helium burning.

Composition of the cross section

1. Radiative capture to the ground state:

   (i) l=1, E1 component.

   (ii) l=2, E2 component.

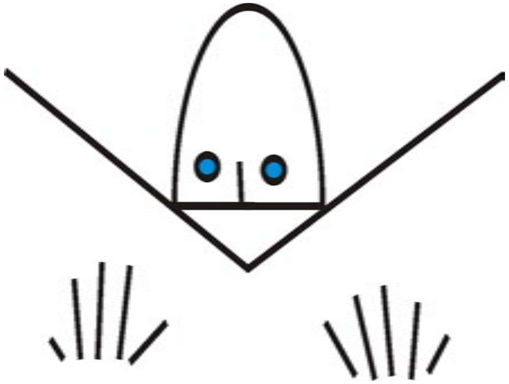
2. Cascade transitions:

   (i) 6.0 state: structurally the same as gs.

   (ii) 6.1 state: l=1,2,3,4,5 possible, but little observed.

   (iii) 6.9 state: l=0,1,2,3,4: 0,2,4 and 13 coherent in total cross section. The strongest cascade.

   (iv) 7.1 state: l=1,2,3,4 possible, little observed.



What do we know?

1. The basic nuclear structure of ^{16}O.
2. Many measurement of the $^{12}C(\alpha,\gamma)^{16}O$ cross section between 1.-3.5 MeV cm.
3. Direct determination of α -widths by scattering methods, like elastics or the ^{16}N spectrum.
4. γ-decay widths
5. More indirect information like from transfer reactions.

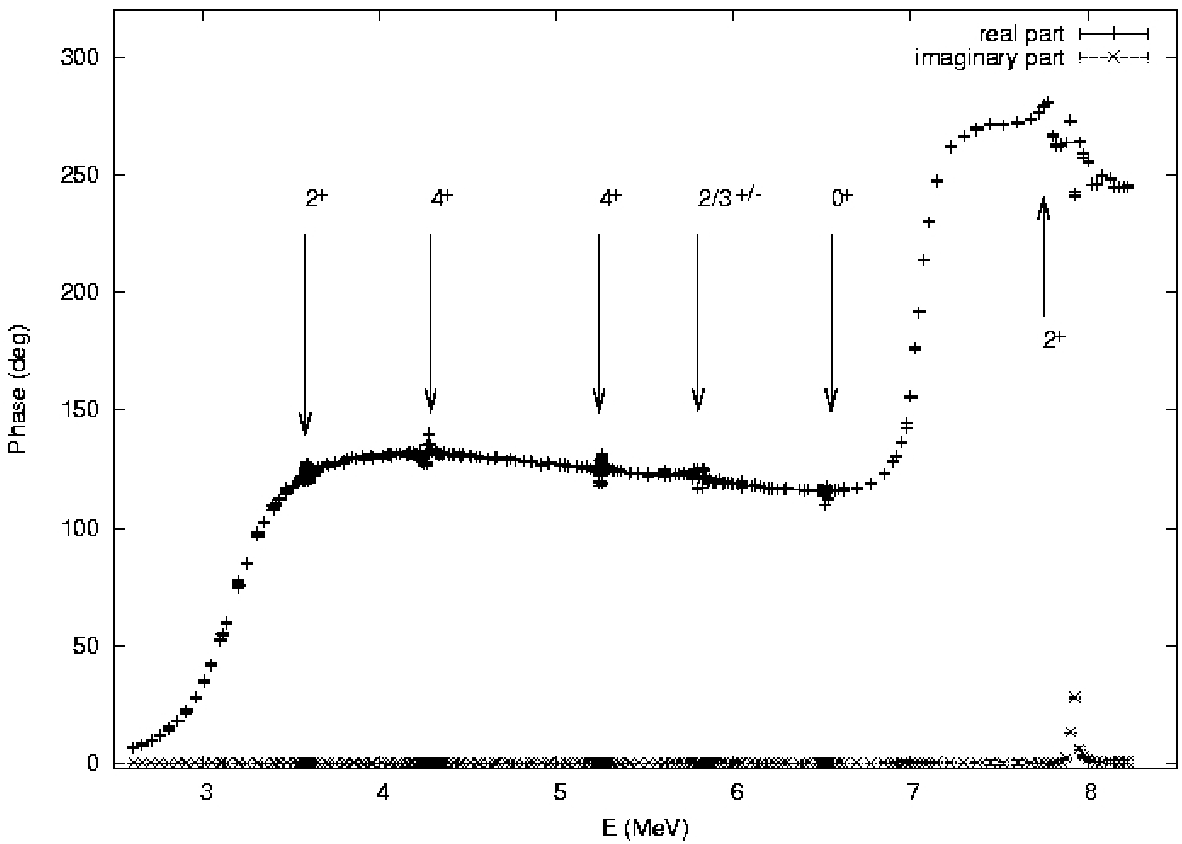
→ **How to bring it all together?**

Data, so called data, and theory

So called data Phaseshifts

l=1 phaseshifts

From Notre Dame data



Electromagnetic transitions

From the Maxwell equations the charge and current free equations for the vector potential **A** can be derived as:

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \qquad \nabla \cdot \mathbf{A} = 0$$

Electric and magnetic fields are given by derivatives of the vector potential:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \qquad \mathbf{H} = \nabla \times \mathbf{A}$$

Looking for stationary solutions and separating by angular momentum gives two solutions for the electromagnetic radiation (`electric' and 'magnetic'):

$$\mathbf{A}_{LM}^{E} = k_\lambda^{-1} C_L \nabla \times \mathbf{L} u_{LM} \qquad \mathbf{A}_{LM}^{M} = i C_L \mathbf{L} u_{LM}$$

with the solutions of the one dimensional Helmholtz equation:

$$u_{LM} = j_L(k_\lambda r) Y_{LM}(\theta,\phi) \approx \frac{(\omega r / c)^L}{(2L+1)!!} Y_{LM}(\theta,\phi)$$

Electromagnetic transitions

With the latter approximation because of the long wavelengths of gamma radiation compared to a nuclear size.

First order perturbation theory gives the transition probability as:

$$T_{i\to f} = \frac{2\pi}{\hbar} |< f \,|H'|\, i >|^2 \frac{dN}{dE}$$

with the hamiltonian *H'*, built upon the vector potential **A.**

$$H' = -\frac{e}{mc}\frac{\mathbf{p}\cdot\mathbf{A}}{2} - \mu_p \frac{e\hbar}{2mc}\sigma\cdot\mathbf{H}$$

After lots of approximations is then:

$$T^{EL}_{i\to f} = O\left[\omega\left(\frac{e^2}{\hbar c}\right)\left(\frac{\omega a}{c}\right)^{2L}\right] \qquad T^{ML}_{i\to f} = O\left[\omega\left(\frac{e^2}{\hbar c}\right)\left(\frac{\hbar}{mca}\right)\left(\frac{\omega a}{c}\right)^{2L}\right]$$

Parity relations: $\Delta\pi \equiv \pi_i/\pi_f = (-1)^L$ E*L} $\quad \Delta\pi \equiv \pi_i/\pi_f = (-1)^{L-1}$ M*L*

Some basics about populating states

angular momenta addition: **c** = **a** + **b**

Eigenfunctions of **c**: $$|c\gamma ab\rangle = \sum_{\alpha\beta} |a\alpha\rangle |b\beta\rangle (a\alpha, b\beta | c\gamma)$$

Clebsch-Gordan coeff.

3 vectors: Racah Coefficients W, 4 vectors 9 j symbols {}.

Normally, we observe many, i.e. an ensemble, of atoms/nuclei being uncorrelated described by an impure state Ψ. An element of the density matrix is then

$$\langle A|\rho|A'\rangle = [\langle A|\Psi\rangle\langle\Psi|A'\rangle]_{avg}$$

for a set of quantum numbers *A* which describe a feature of the system. The expectation value of an operator acting on this system is

$$[Q]_{avg} = \langle\Psi|Q|\Psi\rangle = \sum_{AA'} \langle A|Q|A'\rangle [\langle A'|\Psi\rangle\langle\Psi|A\rangle]_{avg} = Tr[Q\rho]$$

$$Tr[\rho^2] \le 1$$

For the eigenstates |aα> a density tensor can be constructed so that:

$$\rho_{k\kappa}(aa') = \sum_{\alpha\alpha'} (-)^{a'-\alpha'} (a\alpha, a'-\alpha'|k\kappa) \langle a\alpha|\rho|a\alpha'\rangle$$

Example: no polarization

$$\langle a\alpha|\rho|a\alpha'\rangle = \delta_{\alpha\alpha'}/\hat{a}^2$$

and

$$\rho_{00}(aa) = 1/\hat{a}$$

More basics about populating states

Unitarity results in:

$$<a\alpha\,|\,\rho\,|\,a'\alpha'> = \sum_{k\kappa} (-)^{a'-\alpha'} (a\alpha, a'-\alpha'\,|\,k\kappa)\,\rho_{k\kappa}(aa')$$

Similar the efficiency tensor is:

$$\varepsilon_{k\kappa}(aa') = \sum_{\alpha\alpha'} (-)^{a'-\alpha'} (a\alpha, a'-\alpha'\,|\,k\kappa) <a\alpha\,|\,\varepsilon\,|\,a\alpha'>$$

Then is the correlation function is:

$$W = Tr[\rho\varepsilon] = \sum_{aa'\alpha\alpha'} <a\alpha\,|\,\rho\,|\,a'\alpha'><a\alpha\,|\,\varepsilon\,|\,a'\alpha'> = \sum_{aa'k\kappa} \rho_{k\kappa}(aa')\varepsilon_{k\kappa}(aa')$$

Wigner-Eckart theorem: Be $|b\beta>|l\lambda> = |b\beta l\lambda>$ the final product state from:

$$<b\beta l\lambda| = \sum_{a\alpha} <b\beta l\lambda|V|a\alpha><a\alpha|$$

Then there is an interaction matrix element independent of α, β, and λ so that

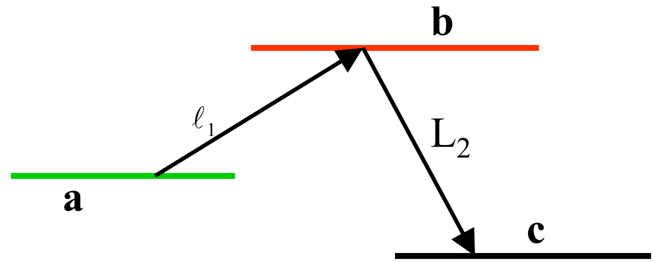
$$<b\beta l\lambda| = <b\,|\,l\,||\,a> \sum_{a\alpha} (b\beta, l\lambda\,|\,a\alpha) <a\alpha|$$

Ground state angular distributions

$\mathbf{a} = \mathbf{s}_1 + \mathbf{s}_2$

$\mathbf{b} = \mathbf{a} + \ell_1$

$\mathbf{b} = \mathbf{c} + \mathbf{L}_2$



Expressions can be derived from e.g. A.J. Ferguson (1965), requiring to work out angular correlation coefficients and keep the accounting right.

For one transition the angular distribution is (very general):

$$W = \sum \rho_{k_b\kappa_b}(bb')\,\rho_{k_{\ell_1}\kappa_{\ell_1}}(\ell_1\ell_1')\,\varepsilon^*_{k_b\kappa_b}(bb')\,\varepsilon^*_{k_{\ell_1}\kappa_{\ell_1}}(\ell_1\ell_1')$$

The density matrices can expressed by the initial state **a**:

$$\rho_{k_b\kappa_b}(bb')\,\rho_{k_{\ell_1}\kappa_{\ell_1}}(\ell_1\ell_1') = \sum \rho_{k\kappa}(aa')(\ldots|)\ldots\{9j\} < b\,\|\, l_1 \,|\, a > < b'\,\|\, l_1' \,|\, a >^*$$

For a particle-γ cascade is then:

$$W(\theta) = \sum (-)^{a-c}(16\pi^2 \hat{a}^2)^{-1}\,\overline{Z}(l_1 b l_1' b'; ak)\,\overline{Z_1}(L_2 b L_2' b'; ck)$$

$$< b\,\|\, l_1 \,|\, a > < b'\,\|\, l_1' \,|\, a >^* < c \,|\, L_2 \,\|\, b > < c \,|\, L_2' \,\|\, b >^* Q_k P_k(\cos\theta)$$

Ground state angular distributions

From angular momenta addition rules the ground state angular distribution in $^{12}C(\alpha,\gamma)^{16}O$ can be derived as

$$W(\theta_\gamma, E) = \frac{1}{1+\frac{\sigma_{E2}}{\sigma_{E1}}}(W_1(\theta_\gamma, E) + W_2(\theta_\gamma, E) + W_{12}(\theta_\gamma, E))$$

with
$$W_1(\theta_\gamma, E) = 1 - P_2(\cos(\theta_\gamma))$$

with
$$W_2(\theta_\gamma, E) = \frac{\sigma(E_2)}{\sigma(E_1)}(1 + \frac{5}{7}P_2(\cos(\theta_\gamma)) - \frac{12}{7}P_4(\cos(\theta_\gamma)))$$

with
$$W_{12}(\theta_\gamma, E) = \frac{6}{5}[5\frac{\sigma(E_2)}{\sigma(E_1)}]^{1/2}\cos(\Phi_{12})(P_1(\cos(\theta_\gamma)) - P_3(\cos(\theta_\gamma)))$$

with
$$\Phi_{12} = \delta_2 - \delta_1 + \arctan(\frac{1}{2}\eta)$$