Chiral Perturbation Theory and Lattice QCD

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2 - 6 December, 2013

I. Standard Model at low energies

1. Interactions

Local symmetries

2. QED+QCD

Precision theory for $E \ll 100\,\text{GeV}$

Qualitative difference QED $\Longleftrightarrow$ QCD

3. Chiral symmetry

Some of the quarks happen to be light

Approximate chiral symmetry

Spontaneous symmetry breakdown

4. Goldstone theorem

If N_f of the quark masses are put equal to zero

QCD contains N_f^2-1 Nambu-Goldstone bosons

5. Gell-Mann-Oakes-Renner relation

Quark masses break chiral symmetry

NGBs pick up mass

M_π^2 is proportional to $m_u + m_d$

II. Chiral perturbation theory

6. Group geometry

Symmetry group of the Hamiltonian G

Symmetry group of the ground state H

Nambu-Goldstone bosons live on G/H

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I. Standard Model at low energies

1. Interactions

strong   weak   e.m.   gravity

$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \; \mathrm{D}$

Gravity

understood only at classical level
gravitational waves $\surd$
quantum theory of gravity ?
classical theory adequate for distances large compared to

$$\ell_{\text{Planck}} \equiv \sqrt{\frac{G\,\hbar}{c^3}} = 1.6 \cdot 10^{-35}\,\text{m}$$

Units

- The constants $c, \hbar, \epsilon_0, k$ can be used to express masses, lengths, times, charges, degrees in energy units.
- mass in energy units: $m = m_{\text{SI}}\, c^2$

  $[m_{\text{SI}}] = 1$ kg, $[m] = 1$ J

  1 eV $= 1.602\ldots 10^{-19}$ J
- length in energy units: $\ell = \dfrac{\ell_{\text{SI}}}{\hbar c}$

  $[\ell_{\text{SI}}] = 1$ m, $[\ell] = 1$ J^{-1}

  1 fm^{-1} $=$ 197.32 MeV
- velocity becomes dimensionless: $v = \dfrac{v_{\text{SI}}}{c}$
- charge also dimensionless: $e = \dfrac{e_{\text{SI}}}{\sqrt{\epsilon_0 \hbar c}}$

$\Rightarrow$ Fine structure constant: $\alpha = \dfrac{e_{\text{SI}}^2}{4\pi\epsilon_0\hbar c} = \dfrac{e^2}{4\pi}$

$\Rightarrow$ Bohr radius: $a_{\text{Bohr}} = \dfrac{4\pi\epsilon_0\hbar^2}{e_{\text{SI}}^2 m_{e\text{SI}}} = \dfrac{4\pi}{e^2 m_e}$

- In energy units, the numerical values of the four constants are $c = \hbar = \epsilon_0 = k = 1$.

- Standard Model is a precision theory for the structure of matter
  - constituents, building blocks: u, d, e
  - held together by strong and electromagnetic interactions: gluons, photons
  - at low energies, weak interaction only generates tiny, calculable corrections

  Briefly discuss the qualitative properties of the weak, electromagnetic and strong interactions at low energies

Weak interaction

- frozen at low energies

$$E \ll M_{\mathsf{W}}\, c^2 \simeq 80\,\mathrm{GeV}$$

- heavy quarks and leptons decay into light ones
- $s, c, b, t, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau$ only contribute indirectly, via quantum fluctuations

Electromagnetic interaction

- Final form of the laws obeyed by the electromagnetic field: J. C. Maxwell 1865

  Royal Society Transactions 155 (1865) 459

  survived relativity and quantum theory, unharmed.

- Schrödinger equation for electrons in an electromagnetic field:

  $$\frac{1}{i}\frac{\partial\psi}{\partial t} - \frac{1}{2m_e^2}(\vec{\nabla} + i\,e\vec{A})^2\psi - e\,\varphi\,\psi = 0$$

  contains the potentials $\vec{A}$, φ

- only $\vec{E} = -\vec{\nabla}\varphi - \dfrac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$

  are of physical significance

- Fock pointed out that the Schrödinger equation is invariant under a group of local transformations: Fock 1926

$$\vec{A}' = \vec{A} + \vec{\nabla} f \ , \quad \varphi' = \varphi - \frac{\partial f}{\partial t} \ , \quad \psi' = e^{-ief}\,\psi$$

  describe the same physical situation as $\vec{A}, \varphi, \psi$

- Weyl termed these gauge transformations
- Equivalence principle of the e.m. interaction:

ψ physically equivalent to $e^{-ief}\,\psi$

- e^{-ief} is unitary 1×1 matrix, $e^{-ief} \in$ U(1)
  $f = f(\vec{x}, t)$ space-time dependent function
- gauge invariance $\Longleftrightarrow$ local U(1) symmetry
  electromagnetic field is gauge field of U(1)
  Weyl 1929
- U(1) symmetry + renormalizability
  fully determine the e.m. interaction

Strong interaction

nuclei = p + n $\sim$ 1930

- Nuclear forces
  Stueckelberg, Yukawa $\sim$ 1935

$V_{e.m.} = -\dfrac{e^2}{4\pi r}$	$V_s = -\dfrac{h^2}{4\pi r}\, e^{-\frac{r}{r_0}}$
$\dfrac{e^2}{4\pi} \simeq \dfrac{1}{137}$	$\dfrac{h^2}{4\pi} \simeq 13$
long range	short range
$r_0 = \infty$	$r_0 = \dfrac{\hbar}{M_\pi c} = 1.4 \cdot 10^{-15}\,\mathrm{m}$
$M_\gamma = 0$	$M_\pi\, c^2 \simeq 140\,\mathrm{MeV}$

- Problem with Yukawa formula:
  p and n are extended objects
  diameter comparable to range of force
  formula only holds for $r \gg$ diameter

- Protons, neutrons composed of quarks

  $\mathrm{p} = uud \qquad \mathrm{n} = udd$

- Quarks carry internal quantum number

  $$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \qquad d = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

  occur in 3 "colours"

- Strong interaction is invariant under local rotations in colour space 1973

  $u' = U \cdot u \qquad d' = U \cdot d$

  $$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix} \in \mathrm{SU}(3)$$

- Can only be so if the strong interaction is also mediated by a gauge field

  gauge field of SU(3) $\Rightarrow$ strong interaction

  Quantum chromodynamics

Comparison of e.m. and strong interaction

	QED	QCD
symmetry	U(1)	SU(3)
gauge field	$\vec{A}, \varphi$	gluon field
particles	photons	gluons
source	charge	colour
coupling constant	e	g

- All charged particles generate e.m. field
- All coloured particles generate gluon field
- Leptons do not interact strongly because they do not carry colour
- Equivalence principle of the strong interaction:

$$U \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ physically equivalent to } \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

2. QED+QCD

Effective theory for $E \ll M_{\rm W}c^2 \simeq 80\,\mathrm{GeV}$

Symmetry	$\mathrm{U(1)} \times \mathrm{SU(3)}$
Lagrangian	QED+QCD

- Dynamical variables:
  gauge fields for photons and gluons
  Fermi fields for leptons and quarks

- Interaction fully determined by group geometry
  Lagrangian contains 2 coupling constants

$$\boxed{e,\ g}$$

- Quark and lepton mass matrices can be brought to diagonal form, eigenvalues real, positive

$$\boxed{m_e,\, m_\mu,\, m_\tau,\, m_u,\, m_d,\, m_s,\, m_c,\, m_b,\, m_t}$$

- Transformation generates vacuum angle

$$\boxed{\theta}$$

- Precision theory for cold matter, atomic structure, solids, ...

  Bohr radius: $$a = \frac{4\pi}{e^2 m_e}$$

- θ breaks CP

  Neutron dipole moment is very small

⇒ strong upper limit, $\theta \simeq 0$

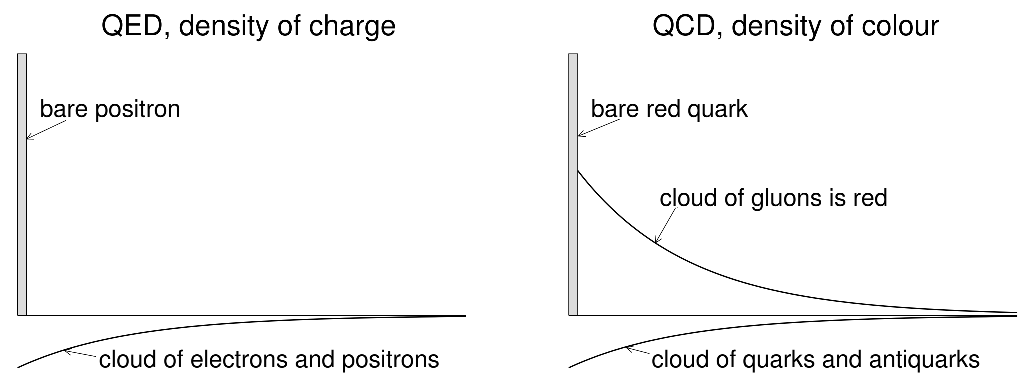
Qualitative difference between e.m. and strong interactions

- Photons do not have charge
- Gluons do have colour

$x_1 \cdot x_2 = x_2 \cdot x_1$ for $x_1, x_2 \in U(1)$ abelian

$x_1 \cdot x_2 \neq x_2 \cdot x_1$ for $x_1, x_2 \in SU(3)$

⇒ Consequence for vacuum polarization



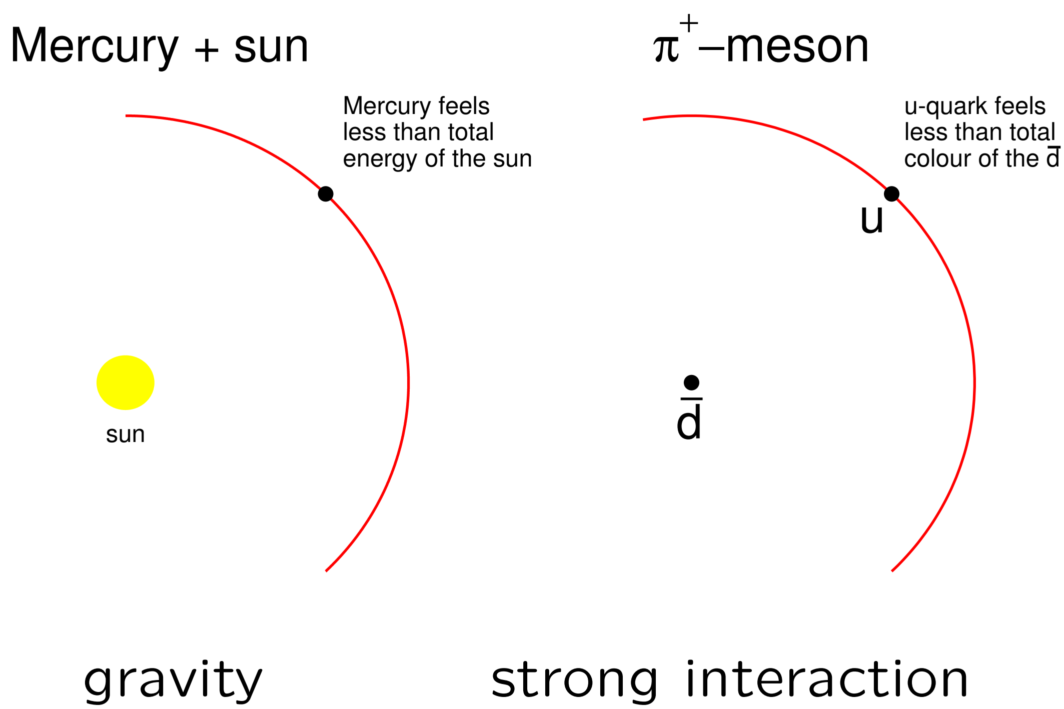
$e < e|_{\text{bare}}$ $\qquad$ $g_s > g_s|_{\text{bare}}$

vacuum shields charge $\qquad$ vacuum amplifies colour

⇒ The electromagnetic and strong interactions polarize the vacuum very differently.

Comparison with gravity

- source of gravitational field: energy
  gravitational field does carry energy
- source of e.m. field: charge
  e.m. field does not carry charge
- source of gluon field: colour
  gluon field does carry colour



Perihelion shift of Mercury:

$$43'' = 50'' - 7'' \text{ per century}$$
$$\Uparrow$$

- Force between u and $\bar{u}$:

$$V_s = -\frac{4}{3}\frac{g^2}{4\pi r}\,, \qquad g \to 0 \quad \text{for} \quad r \to 0$$

$$\frac{g^2}{4\pi} = \frac{6\pi}{(11 N_c - 2N_f)\,|\ln(r\,\Lambda_{\rm QCD})|}$$

$$|\ln(r\,\Lambda_{\rm QCD})| \simeq 7 \quad \text{for} \;\; r = \frac{\hbar}{M_Z\,c} \simeq 2\cdot 10^{-18}\,\text{m}$$

- Vacuum amplifies gluonic field of a bare quark
- Field energy surrounding isolated quark $= \infty$
  Only colour neutral states have finite energy

$\Rightarrow$ Confinement of colour

- $\nexists$ analytic proof that QCD does confine colour. Very good evidence from numerical simulations on a lattice.

QED: interaction weak at low energies
QCD: interaction strong at low energies

$\frac{e^2}{4\pi} \simeq \frac{1}{137}$	$\frac{g^2}{4\pi} \simeq 1$
photons, leptons nearly decouple	gluons, quarks confined

- Nuclear forces = van der Waals forces of QCD

3. Chiral symmetry

- Photons are extremely useful to probe QCD

Much of what we know about the structure of the hadrons stems from scattering experiments involving electrons or photons

$e+N \to e+N$ form factors of the nucleon

$e+N \to e+hadrons$ deep inelastic scattering

electroproduction, photoproduction

- For bound states of quarks,
  e.m. interaction is a small perturbation

  Perturbation series in powers of $\frac{e^2}{4\pi}$ ✓

  Discuss only the leading term: set $e = 0$

- Lagrangian then reduces to QCD

$$\boxed{\; g\,,\, m_u\,,m_d\,,\, m_s\,,\, m_c\,,\, m_b\,,\, m_t}$$

- m_u, m_d, m_s happen to be light

  Consequence:
  Approximate flavour symmetries
  Play a crucial role for the low energy properties

Theoretical paradise

$m_u = m_d = m_s = 0$

$m_c = m_b = m_t = \infty$

QCD with 3 massless quarks

- Lagrangian contains a single parameter: g
  g is net colour of a quark
  depends on radius of the region considered
- Colour contained within radius r

$$\frac{g^2}{4\pi} = \frac{2\pi}{9\,|\ln(r\,\Lambda_{QCD})|}$$

- Intrinsic scale Λ_{QCD} is meaningful,
  but not dimensionless

$\Rightarrow$ No dimensionless free parameter

All dimensionless physical quantities are pure numbers, determined by the theory
Cross sections can be expressed in terms of Λ_{QCD} or in the mass of the proton

- Interactions of u, d, s are identical
  If the masses are set equal to zero,
  there is no difference at all

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

- Lagrangian symmetric under $u \leftrightarrow d \leftrightarrow s$

$$q' = V \cdot q \qquad V \in \mathrm{SU}(3)$$

  V acts on quark flavour, mixes u, d, s

- More symmetry: For massless fermions,
  right and left do not communicate

⇒ Lagrangian of massless QCD is invariant under independent rotations of the right– and left–handed quark fields

$$q_{\mathrm{R}} = \tfrac{1}{2}(1+\gamma_5)\, q\,, \qquad q_{\mathrm{L}} = \tfrac{1}{2}(1-\gamma_5)\, q$$

$$q'_{\mathrm{R}} = V_{\mathrm{R}} \cdot q_{\mathrm{R}} \qquad\qquad q'_{\mathrm{L}} = V_{\mathrm{L}} \cdot q_{\mathrm{L}}$$

$$\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$$

- Massless QCD invariant under $\mathrm{SU}(3)_{\mathrm{R}} \times \mathrm{SU}(3)_{\mathrm{L}}$

  $\mathrm{SU}(3)$ has 8 parameters

$\Rightarrow$ Symmetry under Lie group with 16 parameters

$\Rightarrow$ 16 conserved "charges"

  $Q_1^{\mathrm{V}}, \ldots, Q_8^{\mathrm{V}}$ (vector currents, $R+L$)

  $Q_1^{\mathrm{A}}, \ldots, Q_8^{\mathrm{A}}$ (axial currents, $R-L$)

  commute with the Hamiltonian:

$$[Q_i^{\mathrm{V}}, H_0] = 0 \qquad [Q_i^{\mathrm{A}}, H_0] = 0$$

  "Chiral symmetry" of massless QCD

- Vafa and Witten 1984: state of lowest energy is invariant under the vector charges

  $Q_i^{\mathrm{V}} |0\rangle = 0$

- Axial charges ? $Q_i^{\mathrm{A}} |0\rangle = ?$

Two alternatives for axial charges

$Q_i^{\mathsf{A}}\,\|0\rangle = 0$
Wigner-Weyl realization of G ground state is symmetric $\langle 0\|\,\bar{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,\|0\rangle = 0$ ordinary symmetry spectrum contains parity partners degenerate multiplets of G

$Q_i^{\mathsf{A}}\,\|0\rangle \neq 0$
Nambu-Goldstone realization of G ground state is asymmetric $\langle 0\|\,\bar{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,\|0\rangle \neq 0$ "order parameter" spontaneously broken symmetry spectrum contains Nambu-Goldstone bosons degenerate multiplets of $\mathsf{SU(3)_V} \subset \mathsf{G}$

$$\mathsf{G} = \mathsf{SU(3)_R} \times \mathsf{SU(3)_L}$$

- Spontaneous symmetry breakdown was discovered in condensed matter physics:

  Spontaneous magnetization selects direction

⇒ Rotation symmetry is spontaneously broken

  Nambu-Goldstone bosons = spin waves, magnons

- Nambu 1960: state of lowest energy in particle physics is not invariant under chiral rotations

$$Q_i^{\mathsf{A}}\,|0\rangle \neq 0$$

  For dynamical reasons, the state of lowest energy must be asymmetric

⇒ Chiral symmetry is spontaneously broken

- Very strong experimental evidence √

- Theoretical understanding on the basis of the QCD Lagrangian ?

- Analog of Magnetization ?

$$\bar{q}_\mathsf{R}\, q_\mathsf{L} = \begin{pmatrix} \bar{u}_\mathsf{R}\, u_\mathsf{L} & \bar{d}_\mathsf{R}\, u_\mathsf{L} & \bar{s}_\mathsf{R}\, u_\mathsf{L} \\ \bar{u}_\mathsf{R}\, d_\mathsf{L} & \bar{d}_\mathsf{R}\, d_\mathsf{L} & \bar{s}_\mathsf{R}\, d_\mathsf{L} \\ \bar{u}_\mathsf{R}\, s_\mathsf{L} & \bar{d}_\mathsf{R}\, s_\mathsf{L} & \bar{s}_\mathsf{R}\, s_\mathsf{L} \end{pmatrix}$$

Transforms like $(\bar{3}, 3)$ under $\mathrm{SU}(3)_\mathsf{R} \times \mathrm{SU}(3)_\mathsf{L}$

If the ground state were symmetric, the matrix $\langle 0|\, \bar{q}_\mathsf{R}\, q_\mathsf{L}\, |0\rangle$ would have to vanish, because it singles out a direction in flavour space

"quark condensate", is quantitative measure of spontaneous symmetry breaking
"order parameter"

$\langle 0|\, \bar{q}_\mathsf{R}\, q_\mathsf{L}\, |0\rangle \Leftrightarrow$ magnetization

- Ground state is invariant under $\mathrm{SU}(3)_\mathsf{V}$

$\Rightarrow$ $\langle 0|\, \bar{q}_\mathsf{R}\, q_\mathsf{L}\, |0\rangle$ is proportional to unit matrix

$\langle 0|\, \bar{u}_\mathsf{R}\, u_\mathsf{L}\, |0\rangle = \langle 0|\, \bar{d}_\mathsf{R}\, d_\mathsf{L}\, |0\rangle = \langle 0|\, \bar{s}_\mathsf{R}\, s_\mathsf{L}\, |0\rangle$

$\langle 0|\, \bar{u}_\mathsf{R}\, d_\mathsf{L}\, |0\rangle = \ldots = 0$

4. Goldstone Theorem

- Consequence of $Q_i^{\mathsf{A}}|0\rangle \neq 0$:

$$H_0\, Q_i^{\mathsf{A}}\,|0\rangle = Q_i^{\mathsf{A}}\, H_0\,|0\rangle = 0$$

spectrum must contain 8 states

$$Q_1^{\mathsf{A}}\,|0\rangle, \ldots , Q_8^{\mathsf{A}}\,|0\rangle \qquad \text{with} \quad E = 0,$$

spin 0, negative parity, octet of $\mathsf{SU(3)_V}$
Nambu-Goldstone bosons

- Argument is not water tight:

$$\langle 0|\, Q_i^{\mathsf{A}}\, Q_k^{\mathsf{A}}\,|0\rangle = \int d^3x d^3y\, \langle 0|\, A_i^0(x)\, A_k^0(y)\,|0\rangle$$

$\langle 0|\, A_i^0(x)\, A_k^0(y)\,|0\rangle$ only depends on $\vec{x} - \vec{y}$

$\Rightarrow$ $\langle 0|\, Q_i^{\mathsf{A}}\, Q_k^{\mathsf{A}}\,|0\rangle$ is proportional to the volume of the universe, $|Q_i^{\mathsf{A}}\,|0\rangle| = \infty$

- Rigorous version of Goldstone theorem:
  $\langle 0|\,\bar{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,|0\rangle \neq 0 \Rightarrow \exists$ massless particles

  Proof fasten seatbelts: takes 3 slides

$$Q = \int d^3x\,\bar{u}\gamma^0\gamma_5 d$$
$$[Q, \bar{d}\gamma_5 u] = -\bar{u}u - \bar{d}d$$

- $F^\mu(x-y) \equiv \langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle$
  Lorentz invariance $\Rightarrow$ $F^\mu(z) = z^\mu f(z^2)$
  Chiral symmetry $\Rightarrow$ $\partial_\mu F^\mu(z) = 0$

$$F^\mu(z) = \frac{z^\mu}{z^4} \times \text{constant (for } z^2 \neq 0\text{)}$$

- Spectral decomposition:

$$\langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle =$$
$$\sum_n \langle 0|\bar{u}\gamma^\mu\gamma_5 d|n\rangle\langle n|\,\bar{d}\gamma_5 u\,|0\rangle\; e^{-i\,p_n(x-y)}$$

$p_n^0 \geq 0 \Rightarrow F^\mu(z)$ is analytic in z^0 for $\mathrm{Im}\,z^0 < 0$

$$F^\mu(z) = \frac{z^\mu}{\{(z^0 - i\epsilon)^2 - \vec{z}^{\,2}\}^2} \times \text{constant}$$

- Positive frequency part of massless propagator: (exercise # 1)

$$\Delta^+(z,0) = \frac{i}{(2\pi)^3}\int \frac{d^3p}{2p^0}\, e^{-ipz} \quad , \qquad p^0 = |\vec{p}|$$
$$= \frac{1}{4\pi\, i\,\{(z^0 - i\epsilon)^2 - \vec{z}^{\,2}\}}$$

- Result

$$\boxed{\langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu\Delta^+(z,0)}$$

- Compare Källen–Lehmann representation:

$$\langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle$$
$$= (2\pi)^{-3}\int d^4p\, p^\mu\, \rho(p^2) e^{-ip(x-y)}$$
$$= \int_0^\infty ds\, \rho(s)\partial^\mu\Delta^+(x-y,s)$$

$\Delta^+(z,s) \Longleftrightarrow$ massive propagator

$$\Delta^+(z,s) = \frac{i}{(2\pi)^3}\int d^4p\, \theta(p^0)\, \delta(p^2 - s)\, e^{-ipz}$$

$\Rightarrow$ Only massless intermediate states contribute:

$$\rho(s) = C\,\delta(s)$$

- Why only massless intermediate states ?

  $\langle n|\,\bar{d}\gamma_5 u\,|0\rangle \neq 0$ only if $\langle n|$ has spin 0

  If $|n\rangle$ has spin 0 $\Rightarrow \langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)|n\rangle \propto p^\mu\, e^{-ipx}$

  $\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = 0 \Rightarrow p^2 = 0$

$\Rightarrow$ Either $\exists$ massless particles or $C = 0$

- Claim: $\langle 0|\,\bar{q}_{\mathsf{R}}\, q_{\mathsf{L}}|0\rangle \neq 0 \Rightarrow C \neq 0$

  Lorentz invariance, chiral symmetry

$\Rightarrow \langle 0|\,\bar{d}(y)\gamma_5 u(y)\,\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,|0\rangle = C'\,\partial^\mu\Delta^-(z)$

$\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\,\bar{d}(y)\gamma_5 u(y)]\,|0\rangle$

$$= C\partial^\mu\Delta^+(z,0) - C'\partial^\mu\Delta^-(z,0)$$

- Causality: if $x-y$ is spacelike, then

  $\langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\,\bar{d}(y)\gamma_5 u(y)]\,|0\rangle = 0$

$\Rightarrow C' = -C$

$\Rightarrow \langle 0|\,[\bar{u}(x)\gamma^\mu\gamma_5 d(x),\,\bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C\partial^\mu\Delta(z,0)$

$\Rightarrow \langle 0|\,[\,Q,\,\bar{d}(y)\gamma_5 u(y)]\,|0\rangle = C$

- $\langle 0|\,[\,Q,\,\bar{d}(y)\gamma_5 u(y)]\,|0\rangle = -\langle 0|\bar{u}u + \bar{d}d\,|0\rangle = C$

  Hence $\langle 0|\bar{u}u + \bar{d}d\,|0\rangle \neq 0$ implies $C \neq 0$ qed.

5. Gell-Mann-Oakes-Renner relation

⇒ Spectrum of QCD with 3 massless quarks must contain 8 massless physical particles, $J^P = 0^-$

- Indeed, the 8 lightest mesons do have these quantum numbers:

  $\pi^+, \pi^0, \pi^-, K^+, K^0, \bar{K}^0, K^-, \eta$

  But massless they are not

???

- Real world $\neq$ paradise

  $m_u\,,\,m_d\,,\,m_s \neq 0$

  Quark masses break chiral symmetry,
  allow the left to talk to the right

- Chiral symmetry broken in two ways:

  spontaneously $\qquad \langle 0|\,\bar{q}_{\mathsf{R}}\,q_{\mathsf{L}}\,|0\rangle \neq 0$

  explicitly $\qquad m_u\,,\,m_d\,,\,m_s \neq 0$

- H_{QCD} only has approximate symmetry, to the extent that m_u, m_d, m_s are small

$$H_{\mathrm{QCD}} = H_0 + H_1$$
$$H_1 = \int d^3x \, \{m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s\}$$

- H_0 is Hamiltonian of the massless theory, invariant under $\mathrm{SU(3)_R \times SU(3)_L}$

- H_1 breaks the symmetry, transforms with $(3, \bar{3}) \oplus (\bar{3}, 3)$

- For the low energy structure of QCD, the heavy quarks do not play an essential role: c, b, t are singlets under $\mathrm{SU(3)_R \times SU(3)_L}$ Can include the heavy quarks in H_0

- Nambu-Goldstone bosons are massless only if the symmetry is exact

- Gell-Mann-Oakes-Renner relation:

$$\boxed{M_\pi^2 = (m_u + m_d) \times |\langle 0|\bar{u}\, u\,|0\rangle| \times \frac{1}{F_\pi^2}} \qquad 1968$$

$$\Uparrow \qquad\qquad \Uparrow$$

explicit spontaneous

Coefficient: decay constant F_π

Derivation takes 2 slides

- Pion matrix elements in massless theory:

$$\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-\rangle = i\sqrt{2}\,F\,p^\mu$$
$$\langle 0|\bar{u}\,i\,\gamma_5 d|\pi^-\rangle = \sqrt{2}\,G$$

Only the one–pion intermediate state

$$\langle 0|\bar{u}(x)\gamma^\mu\gamma_5 d(x)\,\bar{d}(y)\gamma_5 u(y)\,|0\rangle = C\,\partial^\mu\Delta^+(z,0)$$
$$\Uparrow$$
$$|\pi^-\rangle\langle\pi^-|$$

contributes. Hence $2\,F\,G = C$

- Value of C fixed by quark condensate

$$C = -\langle 0|\bar{u}u + \bar{d}d\,|0\rangle$$

⇒ Exact result in massless theory:

$$\boxed{F\,G = -\langle 0|\bar{u}u\,|0\rangle}$$

- Matrix elements for $m_{\text{quark}} \neq 0$:

$$\langle 0|\bar{u}\gamma^\mu\gamma_5 d|\pi^-\rangle = i\sqrt{2}\,F_\pi\, p^\mu$$
$$\langle 0|\bar{u}\, i\,\gamma_5 d|\pi^-\rangle = \sqrt{2}\,G_\pi$$

- Current conservation

$$\partial_\mu(\bar{u}\gamma^\mu\gamma_5 d) = (m_u + m_d)\bar{u}\, i\,\gamma_5 d$$
$$\Rightarrow \; F_\pi M_\pi^2 = (m_u + m_d)\, G_\pi$$

$$\boxed{M_\pi^2 = (m_u + m_d)\,\frac{G_\pi}{F_\pi}}$$ exact for $m \neq 0$

- $F_\pi \to F\,, \quad G_\pi \to G$ for $m \to 0$

$$F\,G = -\langle 0|\bar{u}u\,|0\rangle$$

$$\Rightarrow \frac{G_\pi}{F_\pi} = -\frac{\langle 0|\bar{u}u\,|0\rangle}{F_\pi^2} + O(m)$$

$$\Rightarrow M_\pi^2 = (m_u + m_d)\left(\frac{-\langle 0|\bar{u}u\,|0\rangle}{F_\pi^2}\right) + O(m^2) \;\surd$$

$\Rightarrow$ $\langle 0|\bar{u}u\,|0\rangle \leq 0$ if quark masses are positive

- $M_\pi^2 = (m_u + m_d)\,B + O(m^2)$

  $$B = \frac{|\langle 0|\,\bar{u}\,u\,|0\rangle|}{F_\pi^2}\bigg|_{m_u,\,m_d\,\to\,0}$$

- M_π disappears if the symmetry breaking is turned off, $m_u, m_d \to 0$ $\surd$

- Explains why the pseudoscalar mesons have very different masses

  $$M_{K^+}^2 = (m_u + m_s)\,B + O(m^2)$$

  $$M_{K^0}^2 = (m_d + m_s)\,B + O(m^2)$$

$\Rightarrow$ M_K^2 is about 13 times larger than M_π^2, because m_u, m_d happen to be small compared to m_s

- First order perturbation theory also yields

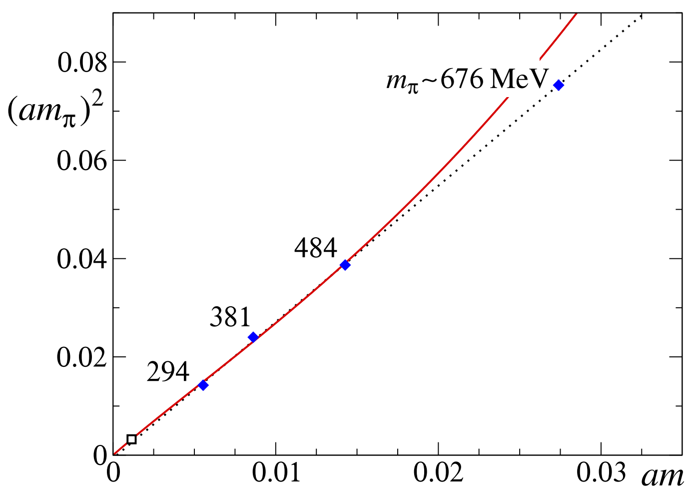
  $$M_\eta^2 = \tfrac{1}{3}\,(m_u + m_d + 4m_s)B + O(m^2)$$

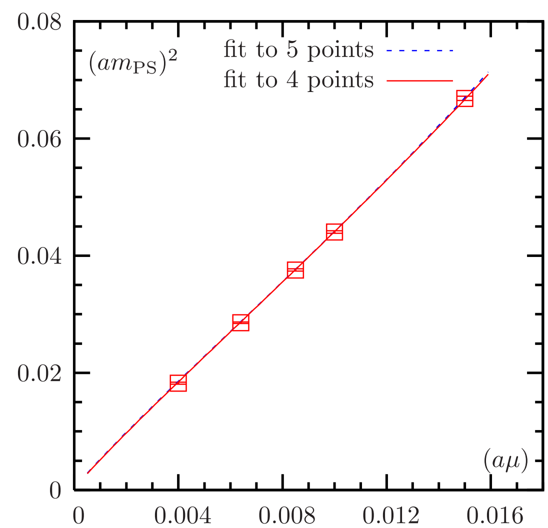
$\Rightarrow$ $M_\pi^2 - 4M_K^2 + 3M_\eta^2 = O(m^2)$

Gell-Mann-Okubo formula for M^2 $\surd$

Checking the GMOR formula on a lattice

- Can determine M_π as function of $m_u = m_d = m$





Lüscher, Lattice conference 2005 ETM collaboration, hep-lat/0701012

- No quenching, quark masses sufficiently light

⇒ Legitimate to use χPT for the extrapolation to the physical values of m_u, m_d

- Quality of data is impressive
- Proportionality of M_π^2 to the quark mass appears to hold out to values of m_u, m_d that are an order of magnitude larger than in nature
- Main limitation: systematic uncertainties

  in particular: $N_f = 2 \rightarrow N_f = 3$

II. Chiral perturbation theory

Scholarpedia: *Chiral Perturbation Theory*

6. Group geometry

- QCD with 3 massless quarks:
  spontaneous symmetry breakdown
  from $\text{SU(3)}_\text{R} \times \text{SU(3)}_\text{L}$ to SU(3)_V
  generates 8 Nambu-Goldstone bosons

- Generalization: suppose symmetry group
  of Hamiltonian is Lie group G
  Generators $Q_1, Q_2, \ldots, Q_D$, $D = \text{dim(G)}$
  For some generators $Q_i\,|0\rangle \neq 0$
  How many Nambu-Goldstone bosons ?

- Consider those elements of the Lie algebra
  $Q = \alpha_1 Q_1 + \ldots + \alpha_n Q_D$, for which $Q\,|0\rangle = 0$
  These elements form a subalgebra:
  $Q\,|0\rangle = 0$, $Q'\,|0\rangle = 0 \;\Rightarrow [Q, Q']\,|0\rangle = 0$
  Dimension of subalgebra: $d \leq D$

- Of the D vectors $Q_i\,|0\rangle$
  $D - d$ are linearly independent

$\Rightarrow$ $D - d$ different physical states of zero mass

$\Rightarrow$ $D - d$ Nambu-Goldstone bosons

- Subalgebra generates subgroup H $\subset$ G
  H is symmetry group of the ground state
  coset space G/H contains as many parameters as there are Nambu-Goldstone bosons
  $d = \dim(\mathrm{H})$, $D = \dim(\mathrm{G})$

$\Rightarrow$ Nambu-Goldstone bosons live on the coset G/H

- Example: QCD with N_f massless quarks
  G $= \mathrm{SU}(N_f)_\mathrm{R} \times \mathrm{SU}(N_f)_\mathrm{L}$
  H $= \mathrm{SU}(N_f)_\mathrm{V}$
  $D = 2\,(N_f^2 - 1)$, $d = N_f^2 - 1$
  $N_f^2 - 1$ Nambu-Goldstone bosons

- It so happens that $m_u, m_d \ll m_s$

- $m_u = m_d = 0$ is an excellent approximation
  $\mathrm{SU}(2)_\mathrm{R} \times \mathrm{SU}(2)_\mathrm{L}$ is a nearly exact symmetry
  $N_f = 2$, $N_f^2 - 1 = 3$ Nambu-Goldstone bosons (pions)

7. Generating functional of QCD

- Basic objects for quantitative analysis of QCD: Green functions of the currents

$$V_a^\mu = \bar{q}\,\gamma^\mu \tfrac{1}{2}\lambda_a\, q\,,\;\; A_a^\mu = \bar{q}\,\gamma^\mu\gamma_5 \tfrac{1}{2}\lambda_a\, q\,,$$
$$S_a = \bar{q}\,\tfrac{1}{2}\lambda_a\, q\,, \qquad P_a = \bar{q}\, i\,\gamma_5\, \tfrac{1}{2}\lambda_a\, q$$

Include singlets, with $\lambda_0 = \sqrt{2/3} \times \mathbf{1}$, as well as

$$\omega = \frac{1}{16\pi^2} \operatorname*{tr}_c G_{\mu\nu}\tilde{G}^{\mu\nu}$$

- Can collect all of the Green functions formed with these operators in a generating functional: Perturb the system with external fields $v_\mu^a(x)$, $a_\mu^a(x)$, $s_a(x)$, $p^a(x)$, $\theta(x)$
Replace the Lagrangian of the massless theory

$$\mathcal{L}_0 = -\frac{1}{2g^2} \operatorname*{tr}_c G_{\mu\nu}G^{\mu\nu} + \bar{q}\, i\gamma^\mu(\partial_\mu - i\, G_\mu)\, q$$

by $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_1 = v_\mu^a V_a^\mu + a_\mu^a A_a^\mu - s^a S_a - p^a P_a - \theta\,\omega$$

- Quark mass terms are included in the external field $s_a(x)$

- $|0\,\text{in}\rangle$: system is in ground state for $x^0 \to -\infty$
  Probability amplitude for finding ground state when $x^0 \to +\infty$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0\,\text{out}|0\,\text{in}\rangle_{v,a,s,p,\theta}$$

- Expressed in terms of ground state of $\mathcal{L}_0$:

$$e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \langle 0|\, T \exp i \int dx \mathcal{L}_1 \,|0\rangle$$

- Expansion of $S_{\text{QCD}}\{v,a,s,p,\theta\}$ in powers of the external fields yields the connected parts of the Green functions of the massless theory

$$S_{\text{QCD}}\{v,a,s,p,\theta\} = -\int dx\, s_a(x) \langle 0|\, S^a(x)\,|0\rangle$$
$$+\frac{i}{2}\int dx dy\, a^a_\mu(x) a^b_\nu(y) \langle 0|\, T A^\mu_a(x) A^\nu_b(y)\,|0\rangle_{\text{conn}} + \ldots$$

- $S_{\text{QCD}}\{v,a,s,p,\theta\}$ is referred to as the *generating functional* of QCD

- For Green functions of full QCD, set

$$s_a(x) = m_a + \tilde{s}_a(x)\,, \quad m_a = \text{tr}\lambda_a\, m$$

  and expand around $\tilde{s}_a(x) = 0$

- Path integral representation for generating functional:

$$\boxed{e^{iS_{\mathsf{QCD}}\{v,a,s,p\}} = \mathcal{N}\int[dG]\, e^{\,i\int dx\,\mathcal{L}_{\mathsf{G}}}\ \det D}$$

$$\mathcal{L}_{\mathsf{G}} = -\frac{1}{2g^2}\,\underset{c}{\mathrm{tr}}\, G_{\mu\nu}G^{\mu\nu} - \frac{\theta}{16\pi^2}\,\underset{c}{\mathrm{tr}}\, G_{\mu\nu}\tilde{G}^{\mu\nu}$$

$$D = i\gamma^{\mu}\{\partial_{\mu} - i(G_{\mu} + v_{\mu} + a_{\mu}\gamma_5)\} - s - i\gamma_5 p$$

G_{μ} is matrix in colour space

v_{μ}, a_{μ}, s, p are matrices in flavour space

$v_{\mu}(x) \equiv \frac{1}{2}\lambda_a\, v^a_{\mu}(x)$, etc.

8. Ward identities

Symmetry in terms of Green functions

- Lagrangian is invariant under

$$q_{\mathsf{R}}(x) \to V_{\mathsf{R}}(x)\, q_{\mathsf{R}}(x)\,, \qquad q_{\mathsf{L}}(x) \to V_{\mathsf{L}}(x)\, q_{\mathsf{L}}(x)$$
$$V_{\mathsf{R}}(x), V_{\mathsf{L}}(x) \in \mathsf{U(3)}$$

provided the external fields are transformed with

$$v'_\mu + a'_\mu = V_{\mathsf{R}}(v_\mu + a_\mu)V_{\mathsf{R}}^\dagger - i\partial_\mu V_{\mathsf{R}} V_{\mathsf{R}}^\dagger$$
$$v'_\mu - a'_\mu = V_{\mathsf{L}}(v_\mu - a_\mu)V_{\mathsf{L}}^\dagger - i\partial_\mu V_{\mathsf{L}} V_{\mathsf{L}}^\dagger$$
$$s' + i\,p' = V_{\mathsf{R}}(s + i\,p)V_{\mathsf{L}}^\dagger$$

The operation takes the Dirac operator into

$$D' = \left\{P_- V_{\mathsf{R}} + P_+ V_{\mathsf{L}}\right\} D \left\{P_+ V_{\mathsf{R}}^\dagger + P_- V_{\mathsf{L}}^\dagger\right\}$$
$$P_\pm = \tfrac{1}{2}(1 \pm \gamma_5)$$

- $\det D$ requires regularization

$\nexists$ symmetric regularization

$\Rightarrow$ $\det D' \neq \det D$, only $|\det D'| = |\det D|$

symmetry does not survive quantization

- Change in $\det D$ can explicitly be calculated

  For an infinitesimal transformation

  $V_{\mathsf{R}} = \mathbf{1} + i\,\alpha + i\beta + \ldots\,, \qquad V_{\mathsf{L}} = \mathbf{1} + i\,\alpha - i\beta + \ldots$

  the change in the determinant is given by

  $$\det D' = \det D\; e^{-i\int dx\,\{2\langle\beta\rangle\omega + \langle\beta\Omega\rangle\}}$$

  $$\langle A\rangle \equiv \mathrm{tr}\,A$$

  $$\omega = \frac{1}{16\pi^2}\,\mathrm{tr}_c\, G_{\mu\nu}\tilde{G}^{\mu\nu} \qquad \text{gluons}$$

  $$\Omega = \frac{N_c}{4\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\partial_\mu v_\nu \partial_\rho v_\sigma + \ldots \qquad \text{ext. fields}$$

- Consequence for generating functional:
  The term with ω amounts to a change in θ,
  can be compensated by $\theta' = \theta - 2\,\langle\beta\rangle$
  Pull term with $\langle\beta\Omega\rangle$ outside the path integral

$$\Rightarrow \boxed{S_{\mathsf{QCD}}\{v',a',s',p',\theta'\} = S_{\mathsf{QCD}}\{v,a,s,p,\theta\} - \int dx \langle\beta\Omega\rangle}$$

$$\boxed{S_{\mathrm{QCD}}\{v',a',s',p',\theta'\} = S_{\mathrm{QCD}}\{v,a,s,p,\theta\} - \int\! dx \langle \beta\Omega\rangle}$$

- S_{QCD} is invariant under $\mathrm{U(3)_R\times U(3)_L}$, except for a specific change due to the anomalies

- Relation plays key role in low energy analysis: collects all of the Ward identities
  For the octet part of the axial current,e.g.

$$\partial^x_\mu \langle 0|\, T A^\mu_a(x)\, P_b(y)\,|0\rangle = -\tfrac{1}{4}\, i\,\delta(x-y)\langle 0|\,\bar{q}\{\lambda_a,\lambda_b\}q\,|0\rangle$$
$$+ \langle 0|\, T\bar{q}(x)\, i\gamma_5\{m,\tfrac{1}{2}\lambda_a\}q(x)\, P_b(y)\,|0\rangle$$

- Symmetry of the generating functional implies the operator relations

$$\partial_\mu V^\mu_a = \bar{q}\, i[m,\tfrac{1}{2}\lambda_a]q\,, \qquad a = 0,\ldots,8$$
$$\partial_\mu A^\mu_a = \bar{q}\, i\gamma_5\{m,\tfrac{1}{2}\lambda_a\}q\,, \qquad a = 1,\ldots,8$$
$$\partial_\mu A^\mu_0 = \sqrt{\tfrac{2}{3}}\,\bar{q}\, i\gamma_5 m q + \sqrt{6}\,\omega$$

- Textbook derivation of the Ward identities goes in inverse direction, but is slippery formal manipulations, anomalies ?

9. Low energy expansion

- If the spectrum has an energy gap

$\Rightarrow$ no singularities in scattering amplitudes or Green functions near $p = 0$

$\Rightarrow$ low energy behaviour may be analyzed with Taylor series expansion in powers of p

$$f(t) = 1 + \tfrac{1}{6}\langle r^2\rangle\, t + \ldots \quad \text{form factor}$$
$$T(p) = a + b\, p^2 + \ldots \quad \text{scattering amplitude}$$

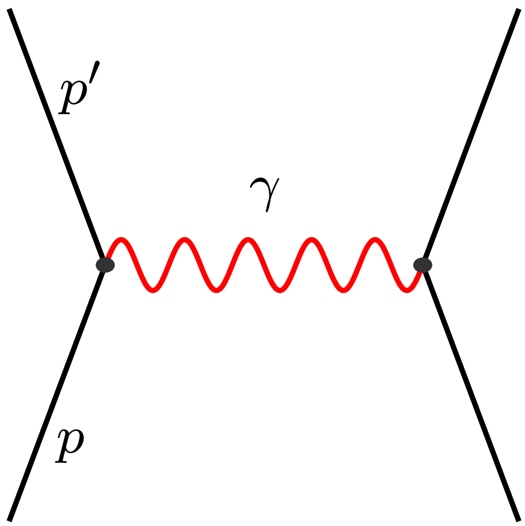
Cross section dominated by S–wave scattering length $\qquad \dfrac{d\sigma}{d\Omega} \simeq |a|^2$

- Expansion parameter: $\dfrac{p}{m} = \dfrac{\text{momentum}}{\text{energy gap}}$

- Taylor series only works if the spectrum has an energy gap, i.e. if there are no massless particles

- Illustration: Coulomb scattering

$$e + e \to e + e$$



Photon exchange $\Rightarrow$ pole at $t = 0$

$$T = \frac{e^2}{(p' - p)^2}$$

Scattering amplitude does not admit Taylor series expansion in powers of p

- QCD does have an energy gap but the gap is very small: M_π

$\Rightarrow$ Taylor series has very small radius of convergence, useful only for $p < M_\pi$

- Massless QCD contains infrared singularities due to the Nambu-Goldstone bosons
- For $m_u = m_d = 0$, pion exchange gives rise to poles and branch points at $p = 0$

$\Rightarrow$ Low energy expansion is not a Taylor series, contains logarithms

Singularities due to Nambu-Goldstone bosons can be worked out with an effective field theory **Chiral Perturbation Theory**

Weinberg, Dashen, Pagels, Gasser, . . .

- Chiral perturbation theory correctly reproduces the infrared singularities of QCD
- Quantities of interest are expanded in powers of external momenta and quark masses
- Expansion has been worked out to next-to-leading order for many quantities "Chiral perturbation theory to one loop"
- In quite a few cases, the next-to-next-to-leading order is also known

- Properties of the Nambu-Goldstone bosons are governed by the hidden symmetry that is responsible for their occurrence
- Focus on the singularities due to the pions

$$H_{\text{QCD}} = H_0 + H_1$$
$$H_1 = \int d^3x \, \{m_u \bar{u} u + m_d \bar{d} d\}$$

H_0 is invariant under $\text{G} = \text{SU}(2)_\text{R} \times \text{SU}(2)_\text{L}$

$|0\rangle$ is invariant under $\text{H} = \text{SU}(2)_\text{V}$

mass term of strange quark is included in H_0

- Treat H_1 as a perturbation

Expansion in powers of H_1	$\Longleftrightarrow$	Expansion in powers of m_u, m_d

- Extension to $\text{SU}(3)_\text{R} \times \text{SU}(3)_\text{L}$ straightforward: include singularities due to exchange of K, η

$\Rightarrow$ Discuss this later, first treat only m_u, m_d as small quantities, keep m_s fixed at the physical value, study the effective theory belonging to $\text{SU}(2)_\text{R} \times \text{SU}(2)_\text{L}$

10. Effective Lagrangian

- Replace quarks and gluons by pions

$$\vec{\pi}(x) = \{\pi^1(x), \pi^2(x), \pi^3(x)\}$$

$$\mathcal{L}_{\text{QCD}} \;\rightarrow\; \mathcal{L}_{eff}$$

- Central claim:

  A. Effective theory yields alternative representation for generating functional of QCD

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{eff}\int[d\pi]e^{i\int dx \mathcal{L}_{eff}\{\vec{\pi},v,a,s,p,\theta\}}}$$

  B. $\mathcal{L}_{eff}$ has the same symmetries as $\mathcal{L}_{\text{QCD}}$

$\Rightarrow$ Can calculate the low energy expansion of the Green functions with the effective theory.
If $\mathcal{L}_{eff}$ is chosen properly, this reproduces the low energy expansion of QCD, order by order.

- Proof of A and B: H.L., Annals Phys. 1994

- Pions live on the coset G/H = SU(2)

$$\vec{\pi}(x) \to U(x) \in \mathsf{SU(2)}$$

The fields $\vec{\pi}(x)$ are the coordinates of $U(x)$

Can use canonical coordinates, for instance

$$U = \exp i\,\vec{\pi}\cdot\vec{\tau} \in \mathsf{SU(2)}$$

- Action of the symmetry group on the quarks:

$$q'_{\mathsf{R}} = V_{\mathsf{R}}\cdot q_{\mathsf{R}}\,, \quad q'_{\mathsf{L}} = V_{\mathsf{L}}\cdot q_{\mathsf{L}}$$

- Action on the pion field:

$$U' = V_{\mathsf{R}}\cdot U\cdot V_{\mathsf{L}}^{\dagger}$$

Note: Transformation law for the coordinates $\vec{\pi}$ is complicated, nonlinear

- Except for the contribution from the anomalies, $\mathcal{L}_{\mathit{eff}}$ is invariant

$$\boxed{\mathcal{L}_{\mathit{eff}}\{U',v',a',s',p',\theta'\} = \mathcal{L}_{\mathit{eff}}\{U,v,a,s,p,\theta\}}$$

Symmetry of S_{QCD} implies symmetry of $\mathcal{L}_{\mathit{eff}}$

Side remark

- For nonrelativistic effective theories, the effective Lagrangian is in general invariant only up to a total derivative.

⇒ From the point of view of effective field theory, nonrelativistic systems with Nambu-Goldstone bosons are more complicated than relativistic ones

detailed discussion: H. L., Phys. Rev. D49 (1994) 3033

- Origin of the complication: the generators of the symmetry group may themselves give rise to order parameters

$$\langle 0|\, Q^i \,|0\rangle \neq 0$$

This cannot happen in the relativistic case:

$$Q = \int d^3x\, j^0(x)$$
$$\langle 0|\, j^\mu(x) \,|0\rangle = 0 \;\Rightarrow \langle 0|\, Q \,|0\rangle = 0$$

Nonrelativistic example where it does happen:

Heisenberg model of a ferromagnet

$$H = -g \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j$$

$g > 0$ $\quad \Uparrow\Uparrow$ lower in energy than $\Uparrow\Downarrow$

- Ground state $= \Uparrow\Uparrow\Uparrow\Uparrow \cdots \Uparrow\Uparrow$
- Magnetization: $\vec{M} = \frac{\mu}{V} \sum_i \vec{s}_i$

$$\langle 0 | \, \vec{M} \, | 0 \rangle \neq 0 \iff \langle 0 | \, \bar{q}_\mathrm{R} \, q_\mathrm{L} \, | 0 \rangle \neq 0$$

- Symmetry generators: $\vec{Q} = \sum_i \vec{s}_i \propto \vec{M}$
- Hamiltonian is invariant under the full rotation group G = SO(3), ground state is invariant only under rotations around the direction of $\langle 0 | \, \vec{M} \, | 0 \rangle$, H = U(1)
- Effective field lives on G/H = S_2: unit vector $\vec{U}$, parametrized by 2 coordinates π^1, π^2.
- Effective Lagrangian of ferromagnet is invariant under local rotations only up to a total derivative. Leading term is related to the Brouwer degree of the map $(\pi^1, \pi^2) \to \vec{U}$.

11. Explicit construction of $\mathcal{L}_{eff}$

- First ignore the external fields,

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, \partial U, \partial^2 U, \dots)$$

Derivative expansion:

$$\mathcal{L}_{eff} = \underset{\substack{\Uparrow \\ O(1)}}{f_0(U)} + \underset{\substack{\Uparrow \\ O(p^2)}}{f_1(U) \times \Box U} + \underset{\substack{\Uparrow \\ O(p^2)}}{f_2(U) \times \partial_\mu U \times \partial^\mu U} + \dots$$

Amounts to expansion in powers of momenta

- Term of $O(1)$: $f_0(U) = f_0(V_\mathrm{R} U V_\mathrm{L}^\dagger)$
  $V_\mathrm{R} = \mathbf{1}\,, \quad V_\mathrm{L} = U \to V_\mathrm{R} U V_\mathrm{L}^\dagger = \mathbf{1}$

$\Rightarrow$ $f_0(U) = f_0(\mathbf{1})$ irrelevant constant, drop it

- Term with $\Box U$: integrate by parts

$\Rightarrow$ can absorb $f_1(U)$ in $f_2(U)$

$\Rightarrow$ Derivative expansion of $\mathcal{L}_{eff}$ starts with

$$\mathcal{L}_{eff} = f_2(U) \times \partial_\mu U \times \partial^\mu U + O(p^4)$$

- Replace the partial derivative by

$$\Delta_\mu \equiv \partial_\mu U U^\dagger , \quad \text{tr}\Delta_\mu = 0$$

Δ_μ is invariant under $\text{SU}(2)_\text{L}$ and transforms with the representation $D^{(1)}$ under $\text{SU}(2)_\text{R}$:

$\Delta_\mu \to V_\text{R} \, \Delta_\mu \, V_\text{R}^\dagger$

In this notation, leading term is of the form

$$\mathcal{L}_{eff} = \tilde{f}_2(U) \times \Delta_\mu \times \Delta^\mu + O(p^4)$$

- Invariance under $\text{SU}(2)_\text{L}$: $\tilde{f}_2(U) = \tilde{f}_2(U V_\text{L}^\dagger)$

$\Rightarrow$ $\tilde{f}_2(U)$ is independent of U

- Invariance under $\text{SU}(2)_\text{R}$: $\Delta_\mu \times \Delta^\mu$ transforms with $D^{(1)} \times D^{(1)} \to$ contains unity exactly once: $\text{tr}(\Delta_\mu \Delta^\mu) = \text{tr}(\partial_\mu U U^\dagger \partial^\mu U U^\dagger) = -\text{tr}(\partial_\mu U \partial^\mu U^\dagger)$

$\Rightarrow$ Geometry fixes leading term up to a constant

$$\boxed{\mathcal{L}_{eff} = \frac{F^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)}$$

$$\boxed{\mathcal{L}_{eff} = \frac{F^2}{4}\text{tr}(\partial_\mu U \partial^\mu U^\dagger) + O(p^4)}$$

- Lagrangian of the nonlinear σ-model
- Expansion in powers of $\vec{\pi}$:

$$U = \exp i\,\vec{\pi}\cdot\vec{\tau} = \mathbf{1} + i\,\vec{\pi}\cdot\vec{\tau} - \tfrac{1}{2}\vec{\pi}^{\,2} + \ldots$$

$\Rightarrow$ $\mathcal{L}_{eff} = \frac{F^2}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{F^2}{48}\text{tr}\{[\partial_\mu\pi,\pi]\,[\partial^\mu\pi,\pi]\} + \ldots$

For the kinetic term to have the standard normalization: rescale the pion field, $\vec{\pi} \to \vec{\pi}/F$

$$\mathcal{L}_{eff} = \tfrac{1}{2}\,\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} + \frac{1}{48F^2}\text{tr}\{[\partial_\mu\pi,\pi]\,[\partial^\mu\pi,\pi]\} + \ldots$$

$\Rightarrow$ a. Symmetry requires the pions to interact

b. Derivative coupling: Nambu-Goldstone bosons only interact if their momentum does not vanish $\Rightarrow$ $\not{\lambda}\pi^4$

- Expression given for $\mathcal{L}_{eff}$ only holds if the external fields are turned off. Also, tr$(\partial_\mu U \partial^\mu U^\dagger)$ is invariant only under global transformations

  - Suffices to replace $\partial_\mu U$ by

$$D_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu)$$

  In contrast to tr$(\partial_\mu U \partial^\mu U^\dagger)$, the term tr$(D_\mu U D^\mu U^\dagger)$ is invariant under local SU(2)$_\mathrm{R}\times$ SU(2)$_\mathrm{L}$

  - Can construct further invariants: $s + ip$ transforms like $U \Rightarrow$ tr$\{(s+ip)U^\dagger\}$ is invariant Violates parity, but tr$\{(s+ip)U^\dagger\}+$tr$\{(s-ip)U\}$ is even under $p \to -p, \vec{\pi} \to -\vec{\pi}$

  - In addition, $\exists$ invariant independent of U: $D_\mu\theta D^\mu\theta$, with $D_\mu\theta = \partial_\mu\theta + 2\,\mathrm{tr}(a_\mu)$

- Count the external fields as $\theta = O(1), \quad v_\mu, a_\mu = O(p), \quad s, p = O(p^2)$

- Derivative expansion yields string of the form

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots$$

- Full expression for leading term:

$$\boxed{\mathcal{L}^{(2)} = \frac{F^2}{4} \langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle + h_0 D_\mu \theta D^\mu \theta}$$

$$\chi \equiv 2\, B\, (s + ip)\,, \qquad \langle X \rangle \equiv \mathrm{tr}(X)$$

- At LO, symmetry allows 2 "low energy constants" (F, B) plus 1 "contact term" (h_0)
- Next-to-leading order:

$$\begin{aligned}\mathcal{L}^{(4)} = &\frac{\ell_1}{4}\langle D_\mu U D^\mu U^\dagger\rangle^2 + \frac{\ell_2}{4}\langle D_\mu U D_\nu U^\dagger\rangle \\ &+ \frac{\ell_3}{4}\langle \chi U^\dagger + U\chi^\dagger\rangle^2 + \frac{\ell_4}{4}\langle D_\mu \chi D^\mu U^\dagger + D_\mu U D^\mu \chi^\dagger\rangle \\ &+ \ldots\end{aligned}$$

- Altogether 7 LEC + 3 CT at NLO
- Number of LEC rapidly grows with the order of the expansion

- Infinitely many LEC
  Symmetry does not determine these
  Predictivity ?
- Essential point: If $\mathcal{L}_{eff}$ is known to given order $\Rightarrow$ can work out low energy expansion of the Green functions to that order Weinberg 1979
- F_π, M_π involve 2 LEC at NLO: ℓ_3, ℓ_4.
- In the $\pi\pi$ scattering amplitude, two further LEC enter at NLO: ℓ_1, ℓ_2.
- Note: effective theory is a quantum field theory
  Need to perform the path integral

$$\boxed{e^{iS_{\text{QCD}}\{v,a,s,p,\theta\}} = \mathcal{N}_{eff}\int [dU] e^{i\int dx \mathcal{L}_{eff}\{U,v,a,s,p,\theta\}}}$$

- Classical theory $\Leftrightarrow$ tree graphs
  Need to include graphs with loops

- Power counting in dimensional regularization:
  Graphs with ℓ loops are suppressed by factor $p^{2\ell}$ as compared to tree graphs

$\Rightarrow$ Leading contributions given by tree graphs
Graphs with one loop contribute at next-to-leading order, etc.

- The leading contribution to S_{QCD} is given by the sum of all tree graphs = classical action:

$$S_{\text{QCD}}\{v,a,s,p,\theta\} = \underset{U(x)}{\text{extremum}} \int dx \, \mathcal{L}_{\mathit{eff}}\{U,v,a,s,p,\theta\}$$

III. Illustrations

12. Some tree level calculations

12.1 Extracting the quark condensate from the generating functional

$$e^{iS_{\mathrm{QCD}}\{v,a,s,p,\theta\}} = \langle 0|\, T \exp i \int dx \mathcal{L}_1 \,|0\rangle$$

$$S_{\mathrm{QCD}}\{v,a,s,p,\theta\} = -\int dx\, s_a(x) \langle 0|\, S^a(x)\,|0\rangle$$
$$+\tfrac{i}{2}\int dx dy\, a^a_\mu(x) a^b_\nu(y) \langle 0|\, T A^\mu_a(x) A^\nu_b(y)\,|0\rangle_{\mathrm{conn}} + \ldots$$

12.2 Condensate in terms of effective theory

- Need the effective action for $v = a = p = \theta = 0$ to first order in s

$\Rightarrow$ classical level of effective theory suffices.

- extremum of the classical action: $U = \mathbf{1}$

$$S^1_{\text{QCD}} = \int dx F^2 B \,\text{tr} s(x)$$

$s(x) = \lambda_a s^a(x)$

- comparison with

$$S^1_{\text{QCD}} = - \int dx \, s_a(x) \langle 0 | \, S^a(x) \, | 0 \rangle$$

$$\boxed{\langle 0 | \bar{u} u \, | 0 \rangle = \langle 0 | \, \bar{d} d \, | 0 \rangle = -F^2 B} \qquad (1)$$

Quark condensate in chiral limit:

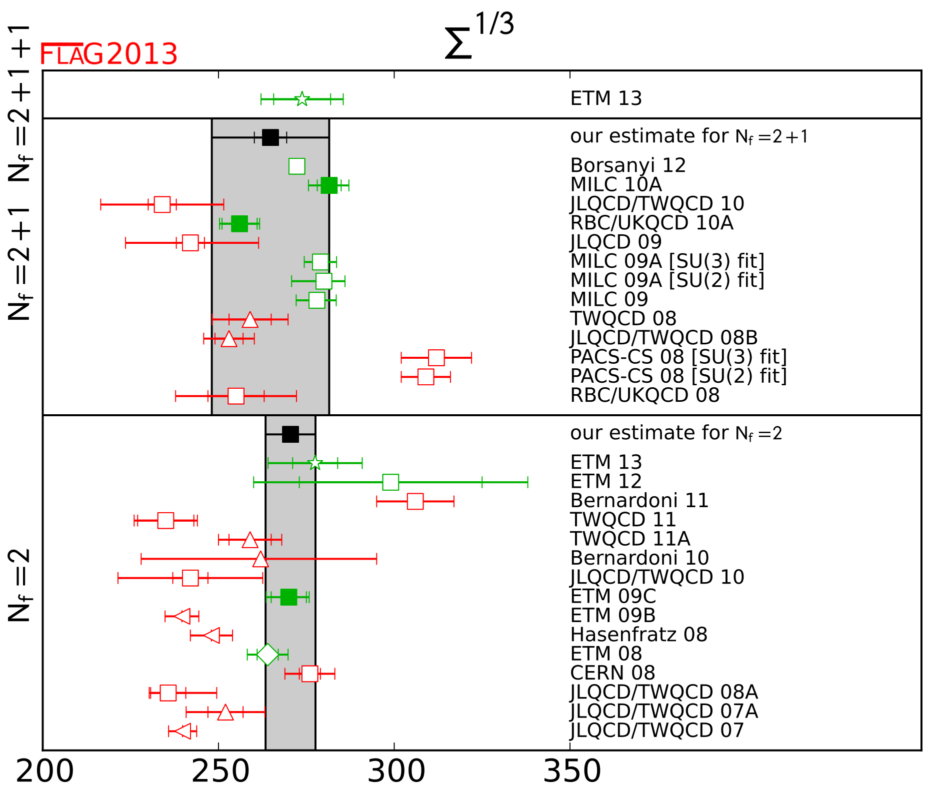
$$\Sigma = |\langle 0|\bar{u}u\,|0\rangle|_{m_u,\,m_d \to 0}$$

$$\Sigma = F^2 B$$

Lattice result (FLAG 2013, arXiv:1310.8555):

$N_f = 2:$ $\Sigma = 270(7)$ MeV

$N_f = 2+1:$ $\Sigma = 265(17)$ MeV



12.3 Evaluation of M_π at tree level

- In classical theory, the square of the mass is the coefficient of the term in the Lagrangian that is quadratic in the meson field:

$$\frac{F^2}{4}\langle \chi U^\dagger + U\chi^\dagger \rangle = \frac{F^2 B}{2}\langle m(U^\dagger + U)\rangle$$
$$= F^2 B(m_u + m_d)\{1 - \frac{\vec{\pi}^{\,2}}{2F^2} + \ldots\}$$

Hence $$\boxed{M_\pi^2 = (m_u + m_d)B} \qquad (2)$$

- Tree level result for F_π:

$$\boxed{F_\pi = F} \qquad (3)$$

- (1) + (2) + (3) $\Rightarrow$ GMOR relation:

$$\boxed{M_\pi^2 = \frac{(m_u + m_d)\,|\langle 0|\bar{u}u\,|0\rangle|}{F_\pi^2}}$$

13. M_π beyond tree level

- The formula $M_\pi^2 = (m_u + m_d)B$ only holds at tree level, represents leading term in expansion of M_π^2 in powers of m_u, m_d
- Disregard isospin breaking: set $m_u = m_d = m$

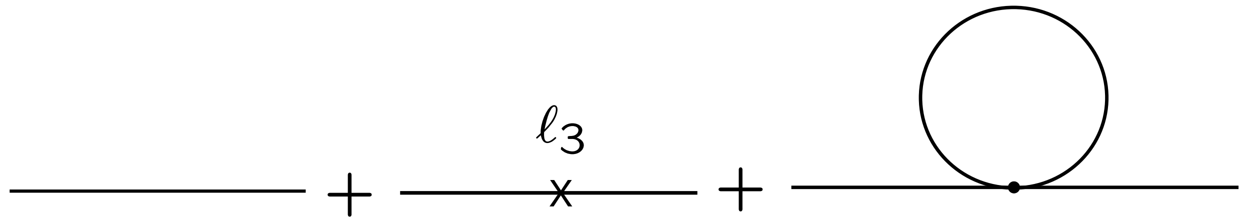
13.1 M_π to 1 loop

- Claim: at next-to-leading order, the expansion of M_π^2 in powers of m contains a logarithm:

$$M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\ln\frac{\Lambda_3^2}{M^2} + O(M^6)$$
$$M^2 \equiv 2mB$$

- Proof: Pion mass $\Leftrightarrow$ pole position, for instance in the Fourier transform of $\langle 0|\, T A_a^\mu(x) A_b^\nu(y)\,|0\rangle$

  Suffices to work out the perturbation series for this object to one loop of the effective theory



- Result (exercise # 5):

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\,\frac{1}{i}\Delta(0,M^2) + O(M^6)$$

$\Delta(0,M^2)$ is the propagator at the origin (exercise # 2):

$$\Delta(0,M^2) = \frac{1}{(2\pi)^d}\int \frac{d^d p}{M^2 - p^2 - i\epsilon}$$
$$= i\,(4\pi)^{-d/2}\,\Gamma(1-d/2) M^{d-2}$$

- Contains a pole at $d = 4$:

$$\Gamma\,(1-d/2) = \frac{2}{d-4} + \ldots$$

- Divergent part is proportional to M^2:

$$M^{d-2} = M^2 \mu^{d-4} (M/\mu)^{d-4} = M^2 \mu^{d-4} e^{(d-4)\ln(M/\mu)}$$
$$= M^2 \mu^{d-4}\{1 + (d-4)\ln(M/\mu) + \ldots\}$$

- Denote the singular factor by

$$\lambda \equiv \frac{1}{2}(4\pi)^{-d/2}\,\Gamma(1-d/2)\,\mu^{d-4}$$
$$= \frac{\mu^{d-4}}{16\pi^2}\left\{\frac{1}{d-4} - \frac{1}{2}(\ln 4\pi + \Gamma'(1) + 1) + O(d-4)\right\}$$

- The propagator at the origin then becomes

$$\frac{1}{i}\Delta(0,M^2) = M^2\left\{2\lambda + \frac{1}{16\pi^2}\ln\frac{M^2}{\mu^2} + O(d-4)\right\}$$

- In the expression for M_π^2

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\,\frac{1}{i}\Delta(0,M^2) + O(M^6)$$

the divergence can be absorbed in ℓ_3:

$$\ell_3 = -\frac{1}{2}\lambda + \ell_3^{\text{ren}}$$

- ℓ_3^{ren} depends on the renormalization scale μ

$$\ell_3^{\text{ren}} = \frac{1}{64\pi^2}\ln\frac{\mu^2}{\Lambda_3^2} \quad \text{running low energy constant}$$

- Λ_3 is the ren. group invariant scale of ℓ_3

- Net result for M_π^2

$$\boxed{M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\ \ln\frac{\Lambda_3^2}{M^2} + O(M^6)}$$

⇒ M_π^2 contains a chiral logarithm at NLO

- Crude estimate for Λ_3, based on SU(3) mass formulae for the pseudoscalar octet:

$$0.2\ \text{GeV} < \Lambda_3 < 2\ \text{GeV}$$

$$\bar{\ell}_3 \equiv \ln\frac{\Lambda_3^2}{M_\pi^2} = 2.9 \pm 2.4$$

Gasser, L. 1984

∃ better determination $\bar{\ell}_3$ on the lattice, to be discussed later

⇒ Next–to–leading term is small correction:

$$0.005 < \frac{1}{2}\,\frac{M_\pi^2}{(4\pi F_\pi)^2}\ \ln\frac{\Lambda_3^2}{M_\pi^2} < 0.040$$

- Scale of the expansion is set by size of pion mass in units of decay constant:

$$\frac{M^2}{(4\pi F)^2} \simeq \frac{M_\pi^2}{(4\pi F_\pi)^2} = 0.0144$$

13.2 M_π to 2 loops

- Terms of order m_{quark}^3:

$$M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\,\ln\frac{\Lambda_3^2}{M^2} + \frac{17}{18}\,\frac{M^6}{(4\pi F)^4}\left(\ln\frac{\Lambda_{\text{M}}^2}{M^2}\right)^2 + k_{\text{M}} M^6 + O(M^8)$$

  F is pion decay constant for $m_u = m_d = 0$
  ChPT to two loops   Colangelo 1995

- Coefficients $\frac{1}{2}$ and $\frac{17}{18}$ determined by symmetry

- $\Lambda_3, \Lambda_{\text{M}}$ and $k_{\text{M}} \Longleftrightarrow$ LEC in $\mathcal{L}_{eff}$

14. F_π to one loop

- Also contains a logarithm at NLO:

$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\Lambda_4^2} + O(M^4)\right\}$$
$$M_\pi^2 = M^2\left\{1 + \frac{M^2}{32\pi^2F^2}\ln\frac{M^2}{\Lambda_3^2} + O(M^4)\right\}$$

F is pion decay constant in limit $m_u, m_d \to 0$

- Structure is the same, coefficients and scale of logarithm are different

- Low energy theorem: at leading order in the chiral expansion, the *scalar radius* is also determined by the scale Λ_4:

$$\langle r^2\rangle_s = \frac{6}{(4\pi F)^2}\left\{\ln\frac{\Lambda_4^2}{M^2} - \frac{13}{12} + O(M^2)\right\}$$

Chiral symmetry relates F_π to $\langle r^2\rangle_s$

What is the scalar radius ? ⇒ next section

15. Pion form factors

- Scalar form factor of the pion:

$$F_s(t) = \langle \pi(p') | \bar{q}\, q \,| \pi(p) \rangle \;\; , \quad t = (p' - p)^2$$

- Definition of scalar radius:

$$F_s(t) = F_s(0) \left\{ 1 + \frac{1}{6} \langle r^2 \rangle_s \, t + O(t^2) \right\}$$

- Low energy theorem:

$$\langle r^2 \rangle_s = \frac{6}{(4\pi F)^2} \left\{ \ln \frac{\Lambda_4^2}{M^2} - \frac{13}{12} + O(M^2) \right\}$$

$\Rightarrow$ In massless QCD, the scalar radius diverges, because the density of the pion cloud only decreases with a power of the distance

- Same infrared singularity also occurs in the charge radius (e.m. current), but coefficient of the chiral logarithm is 6 times smaller:

$$\langle r^2 \rangle_s = \frac{6}{(4\pi F)^2} \left\{ \ln \frac{\Lambda_4^2}{M^2} - \frac{13}{12} + O(M^2) \right\}$$

$$\langle r^2 \rangle_{em} = \frac{1}{(4\pi F)^2} \left\{ \ln \frac{\Lambda_6^2}{M^2} - 1 + O(M^2) \right\}$$

$\Rightarrow$ $\langle r^2 \rangle_s > \langle r^2 \rangle_{em}$ if M small enough

- $\langle r^2 \rangle_{em}$ can be determined experimentally

$$\langle r^2 \rangle_{em} = 0.439 \pm 0.008\,\mathrm{fm}^2$$

  NA7 Collaboration, NP B277 (1986) 168

- Scalar form factor of the pion can be calculated by means of dispersion theory

- Result for the slope:

$$\langle r^2 \rangle_s = 0.61 \pm 0.04\,\mathrm{fm}^2$$

Colangelo, Gasser, L., Nucl. Phys. 2001

⇒ Corresponding value of the scale Λ_4:

$$\Lambda_4 = 1.26 \pm 0.14\,\mathrm{GeV}$$

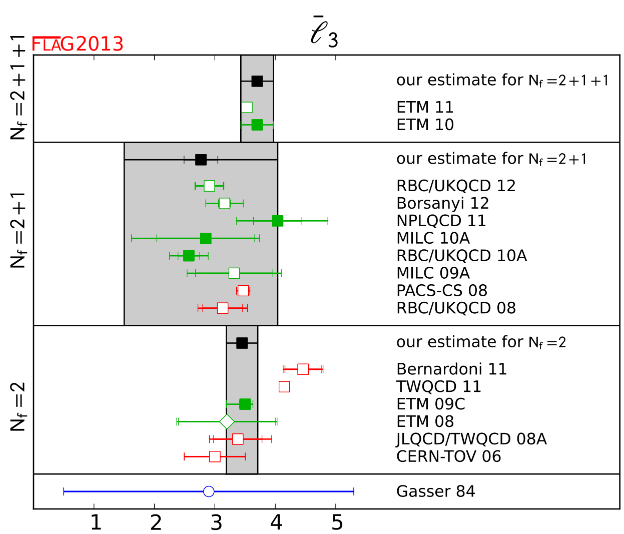
16. Lattice results for M_π, F_π

16.1 Results for M_π

- Determine the scale Λ_3 by comparing the lattice results for M_π as function of m with the χPT formula

$$M_\pi^2 = M^2 - \frac{1}{2}\,\frac{M^4}{(4\pi F)^2}\,\ln\frac{\Lambda_3^2}{M^2} + O(M^6)$$

$$M^2 \equiv 2Bm$$



lattice results for $\bar{\ell}_3$

Horizontal axis shows the value of $\bar{\ell}_3 \equiv \ln\frac{\Lambda_3^2}{M_\pi^2}$

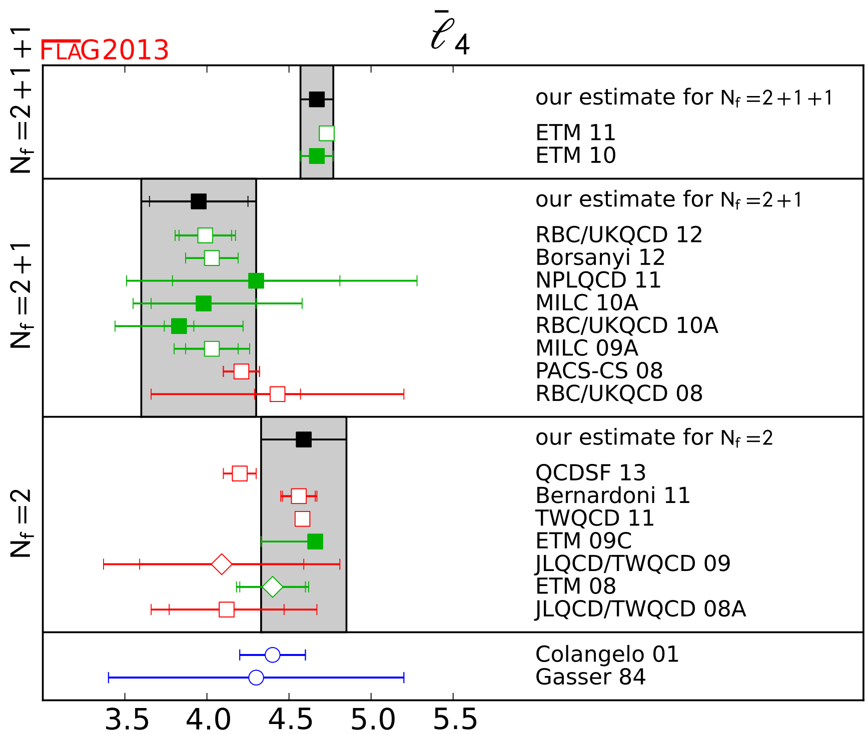
Range for Λ_3 obtained in 1984 corresponds to $\bar{\ell}_3 = 2.9 \pm 2.4$

	$N_f = 2$	$N_f = 2+1$	$N_f = 2+1+1$
$\bar{\ell}_3$	3.45 ± 0.26	2.77 ± 1.27	3.70 ± 0.27

FLAG 2013

16.2 Results for F_π

$$F_\pi = F\left\{1 - \frac{M^2}{16\pi^2F^2}\ln\frac{M^2}{\Lambda_4^2} + O(M^4)\right\}$$



Horizontal axis shows the value of $\bar{\ell}_4 \equiv \ln\frac{\Lambda_4^2}{M_\pi^2}$

- Lattice results beautifully confirm the prediction for the sensitivity of F_π to m_u, m_d:

$$\frac{F_\pi}{F} = 1.072 \pm 0.007$$ Colangelo, Dürr 2004

17. $\pi\pi$ scattering

17.1 Low energy scattering of pions

- Consider scattering of pions with $\vec{p} = 0$
- At $\vec{p} = 0$, only the S-waves survive (angular momentum barrier). Moreover, these reduce to the scattering lengths
- Bose statistics: S-waves cannot have $I = 1$, either have $I = 0$ or $I = 2$

⇒ At $\vec{p} = 0$, the $\pi\pi$ scattering amplitude is characterized by two constants: a_0^0, a_0^2

- Chiral symmetry suppresses the interaction at low energy: Nambu-Goldstone bosons of zero momentum do not interact

⇒ a_0^0, a_0^2 disappear in the limit $m_u, m_d \to 0$

⇒ $a_0^0, a_0^2 \sim M_\pi^2$ measure symmetry breaking

17.2 Tree level of χPT

- Low Energy theorem Weinberg 1966:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} + O(M_\pi^4)$$
$$a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} + O(M_\pi^4)$$

⇒ Chiral symmetry predicts a_0^0, a_0^2 in terms of F_π

- Accuracy is limited: Low energy theorem only specifies the first term in the expansion in powers of the quark masses
  Corrections from higher orders ?

17.3 Scattering lengths at 1 loop

- Next term in the chiral perturbation series:

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left\{1 + \frac{9}{2}\frac{M_\pi^2}{(4\pi F_\pi)^2}\ln\frac{\Lambda_0^2}{M_\pi^2} + O(M_\pi^4)\right\}$$

- Coefficient of chiral logarithm unusually large
  Strong, attractive final state interaction

- Scale Λ_0 is determined by the LEC of $\mathcal{L}_{eff}^{(4)}$:

$$\frac{9}{2}\ln\frac{\Lambda_0^2}{M_\pi^2} = \frac{20}{21}\bar{\ell}_1 + \frac{40}{21}\bar{\ell}_2 - \frac{5}{14}\bar{\ell}_3 + 2\,\bar{\ell}_4 + \frac{5}{2}$$

- Information about $\bar{\ell}_1, \ldots, \bar{\ell}_4$?

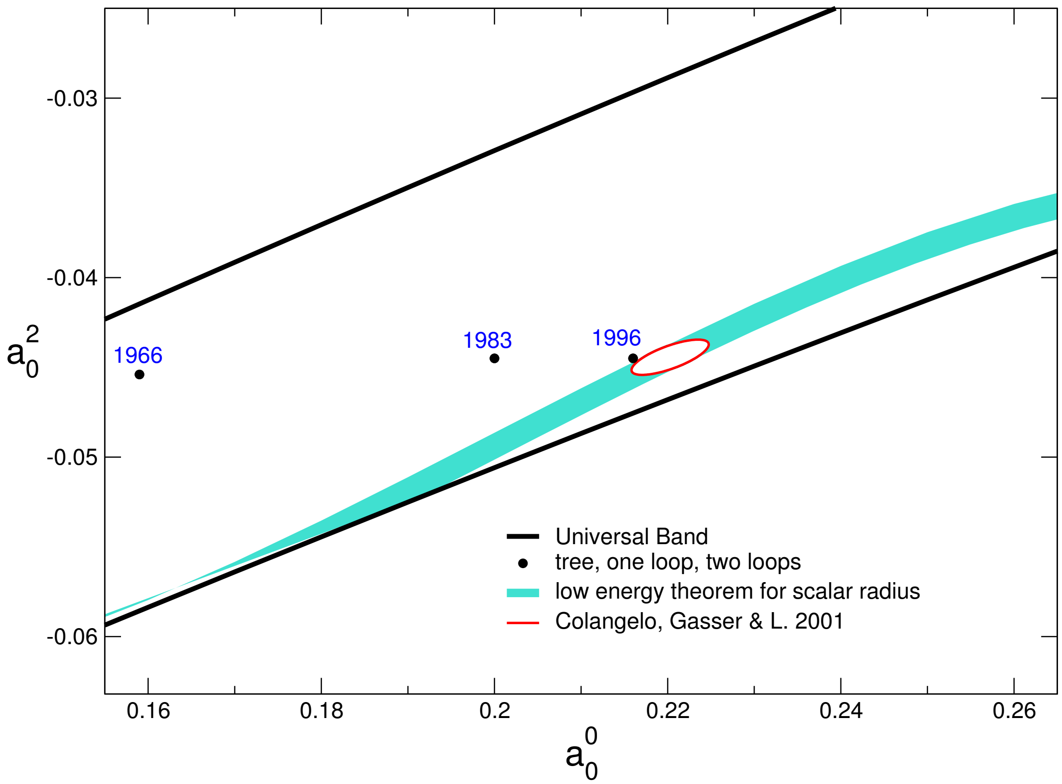
$\bar{\ell}_1, \bar{\ell}_2 \Longleftrightarrow$	momentum dependence of scattering amplitude

$\Rightarrow$ Can be determined phenomenologically

$\bar{\ell}_3, \bar{\ell}_4 \Longleftrightarrow$	dependence of scattering amplitude on quark masses

Have discussed their values already

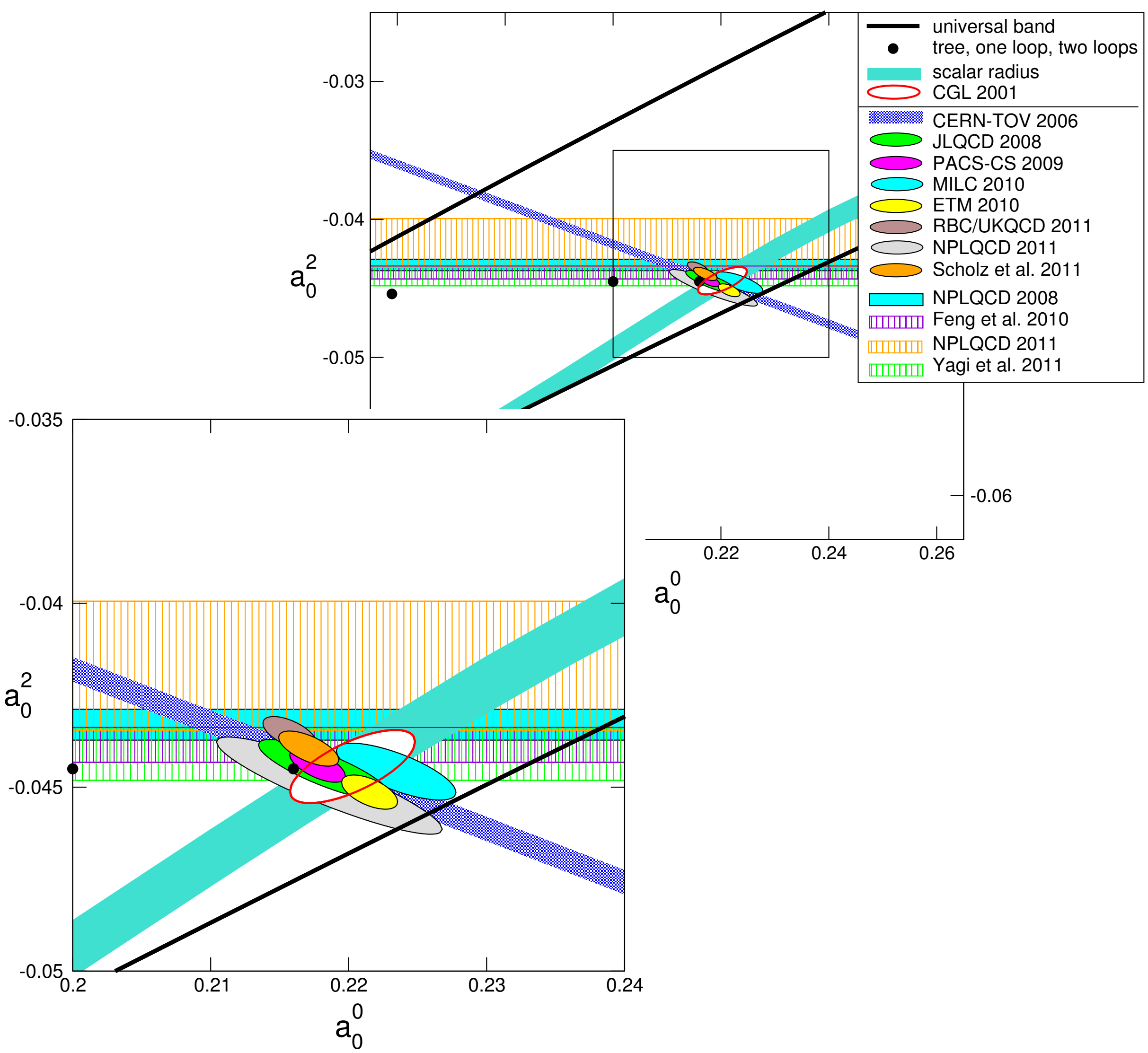
17.4 Numerical predictions from χPT



Sizable corrections in a_0^0 | a_0^2 nearly stays put

17.5 a_0^0, a_0^2 from lattice results for ℓ_3, ℓ_4

- Uncertainty in prediction for a_0^0, a_0^2 is dominated by the uncertainty in the LEC ℓ_3, ℓ_4
- Can make use of the lattice results for these



17.6 Experiments concerning a_0^0, a_0^2

- Production experiments $\pi N \to \pi\pi N$,
  $\psi \to \pi\pi\omega$, $B \to D\pi\pi$, . . .

  Problem: pions are not produced in vacuo

$\Rightarrow$ Extraction of $\pi\pi$ scattering amplitude is not simple

  Accuracy rather limited

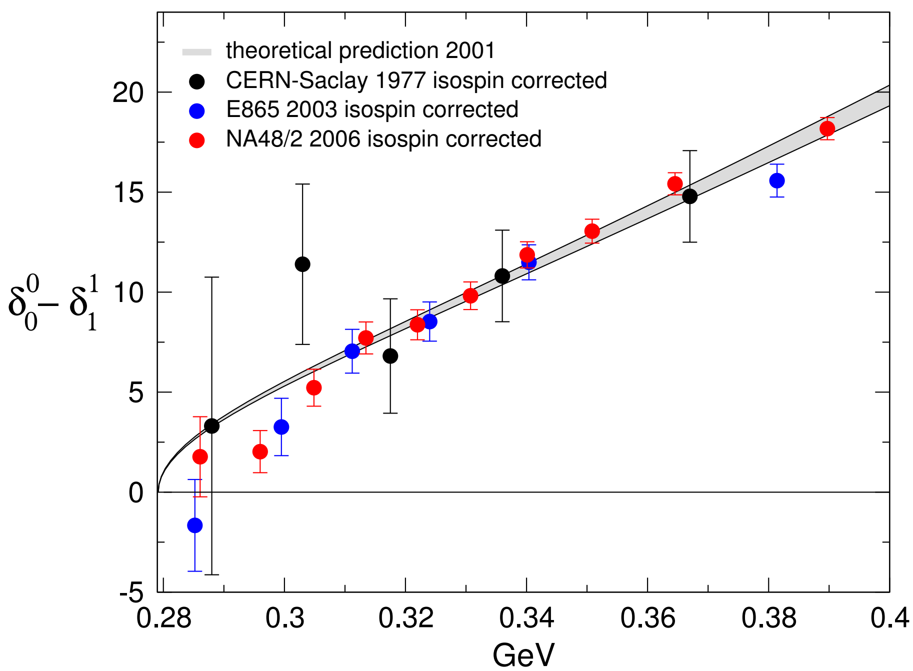
- $K^\pm \to \pi^+\pi^- e^\pm \nu$ data:
  CERN-Saclay, E865, NA48/2
- $K^\pm \to \pi^0\pi^0\pi^\pm$, $K^0 \to \pi^0\pi^0\pi^0$: cusp near threshold, NA48/2
- $\pi^+\pi^-$ atoms, DIRAC

17.7 Results from K_{e4} decay

$K^{\pm} \to \pi^{+}\pi^{-}e^{\pm}\nu$

- Allows clean measurement of $\delta_0^0 - \delta_1^1$

  Theory predicts $\delta_0^0 - \delta_1^1$ as function of energy



Prediction: $a_0^0 = 0.220 \pm 0.005$

NA48/2: $a_0^0 = 0.2206 \underset{stat}{\pm 0.0049} \underset{syst}{\pm 0.0018} \underset{theo}{\pm 0.0064}$

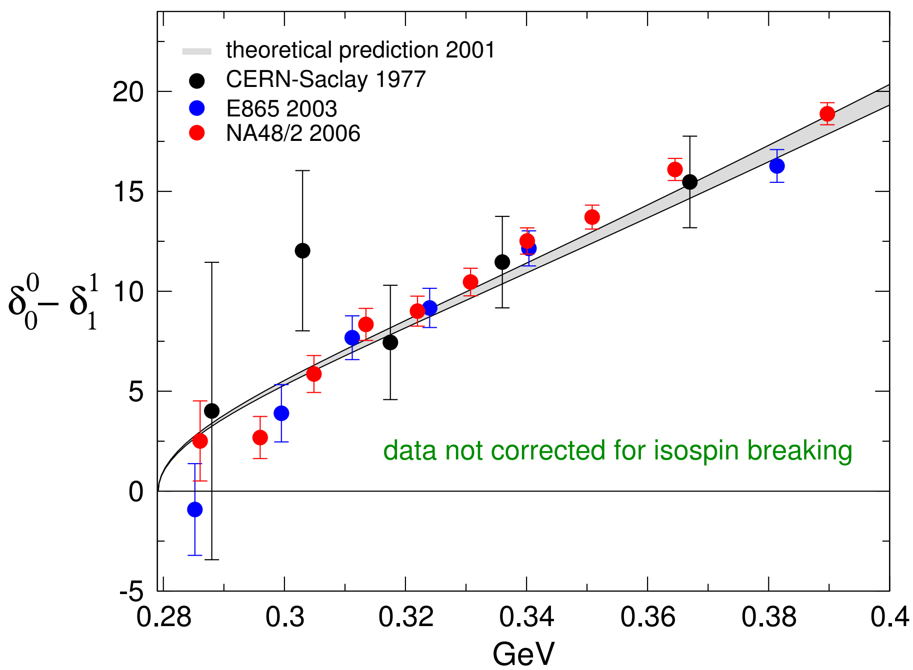
Bloch-Devaux, Chiral Dynamics 2009

- There was a discrepancy here, because a pronounced isospin breaking effect from

$$K \to \pi^0\pi^0 e\nu \to \pi^+\pi^- e\nu$$

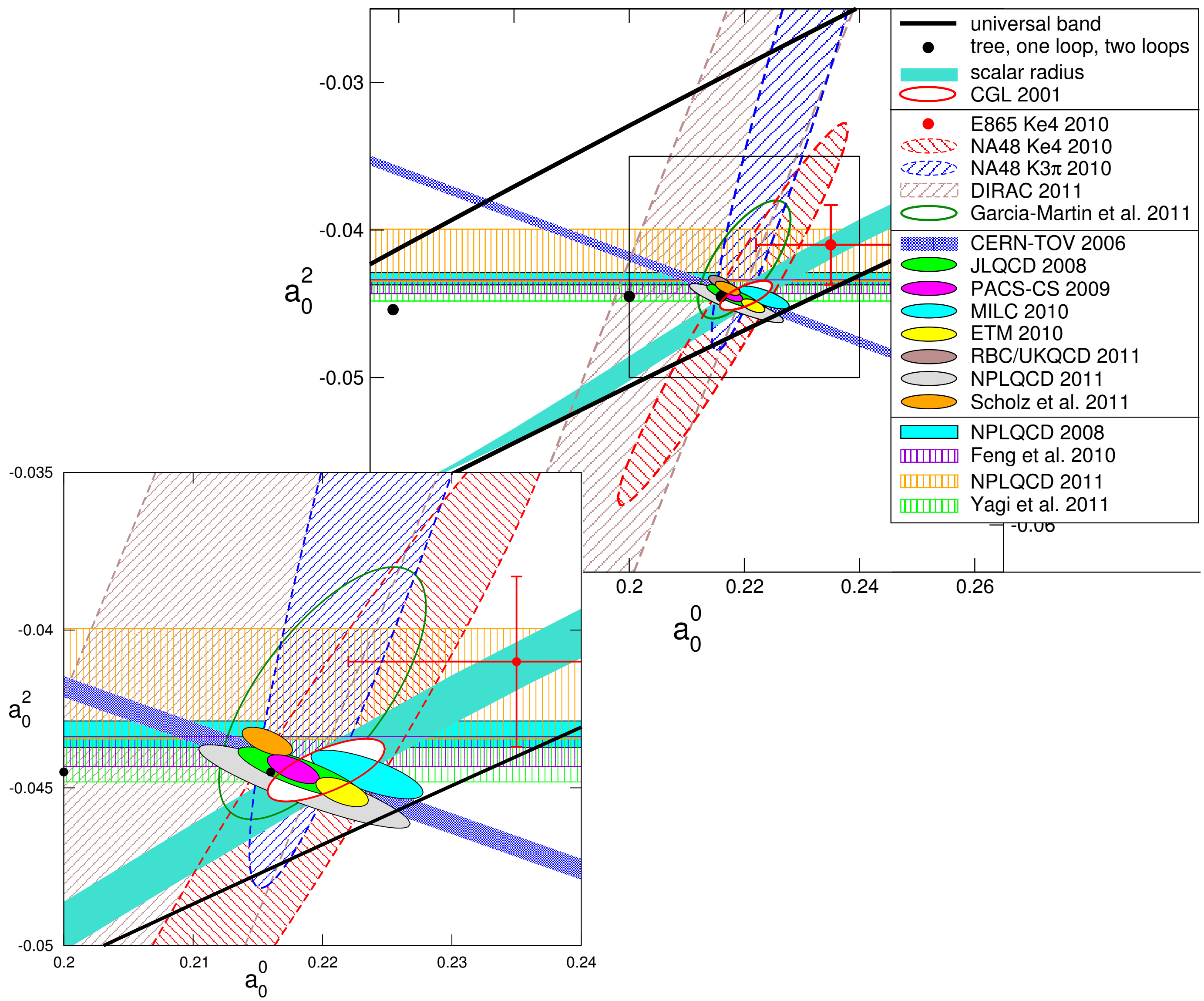
had not been accounted for in the data analysis

Colangelo, Gasser, Rusetsky 2007, Bloch-Devaux 2007



- The correction is not enormous, but matters: If a_0^0 is determined from the uncorrected NA48 data, the central value comes out higher than the theoretical prediction by about 4 times the uncertainty attached to this prediction.

17.8 Summary for a_0^0, a_0^2



18. Conclusions for SU(2)×SU(2)

- Expansion in powers of m_u, m_d yields a very accurate low energy representation of QCD
- Lattice results confirm the GMOR relation

⇒ M_π is dominated by the contribution from the quark condensate

⇒ Energy gap of QCD is understood very well

- Lattice approach allows an accurate measurement of the low energy constant ℓ_3 already now
- Even for ℓ_4, the lattice starts becoming competitive with dispersion theory

Exercises

1. Evaluate the positive frequency part of the massless propagator

$$\Delta^+(z,0) = \frac{i}{(2\pi)^3}\int \frac{d^3k}{2k^0}\, e^{-ikz} \quad , \qquad k^0 = |\vec{k}|$$

for $\mathrm{Im}\, z^0 < 0$. Show that the result can be represented as

$$\Delta^+(z,0) = \frac{1}{4\pi i z^2}$$

2. Evaluate the d-dimensional propagator

$$\Delta(z,M) = \int \frac{d^dk}{(2\pi)^d} \frac{e^{-ikz}}{M^2 - k^2 - i\epsilon}$$

at the origin and verify the representation

$$\Delta(0,M) = \frac{i}{4\pi}\Gamma\left(1-\frac{d}{2}\right)\left(\frac{M^2}{4\pi}\right)^{\frac{d}{2}-1}$$

How does this expression behave when $d \to 4$?

3. Leading order effective Lagrangian:

$$\begin{aligned}
\mathcal{L}^{(2)} &= \frac{F^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle + h_0 D_\mu\theta D^\mu\theta \\
D_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + i\,U(v_\mu - a_\mu) \\
\chi &= 2\,B\,(s + ip) \\
D_\mu\theta &= \partial_\mu\theta + 2\langle a_\mu\rangle \\
\langle X\rangle &= \mathrm{tr}X
\end{aligned}$$

- Take the space-time independent part of the external field $s(x)$ to be isospin symmetric (i. e. set $m_u = m_d = m$):

$$s(x) = m\,\mathbf{1} + \tilde{s}(x)$$

- Expand $U = \exp i\,\phi/F$ in powers of $\phi = \vec{\phi}\cdot\vec{\tau}$ and check that, in this normalization of the field ϕ, the kinetic part takes the standard form

$$\mathcal{L}^{(2)} = \tfrac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \tfrac{1}{2}M^2\vec{\phi}^2 + \ldots$$

  with $M^2 = 2mB$.

- Draw the graphs for all of the interaction vertices containing up to four of the fields $\phi, v_\mu, a_\mu, \tilde{s}, p, \theta$.

4. Show that the classical field theory belonging to the QCD Lagrangian in the presence of external fields is invariant under

$$
\begin{aligned}
v'_\mu + a'_\mu &= V_{\mathsf{R}}(v_\mu + a_\mu)V_{\mathsf{R}}^\dagger - i\partial_\mu V_{\mathsf{R}} V_{\mathsf{R}}^\dagger \\
v'_\mu - a'_\mu &= V_{\mathsf{L}}(v_\mu - a_\mu)V_{\mathsf{L}}^\dagger - i\partial_\mu V_{\mathsf{L}} V_{\mathsf{L}}^\dagger \\
s' + i\,p' &= V_{\mathsf{R}}(s + i\,p)V_{\mathsf{L}}^\dagger \\
q'_{\mathsf{R}} &= V_{\mathsf{R}}\, q_{\mathsf{R}}(x) \\
q'_{\mathsf{L}} &= V_{\mathsf{L}}\, q_{\mathsf{L}}
\end{aligned}
$$

where $V_{\mathsf{R}}, V_{\mathsf{L}}$ are space-time dependent elements of U(3).

5. Evaluate the pion mass to NLO of χPT . Draw the relevant graphs and verify the representation

$$M_\pi^2 = M^2 + \frac{2\,\ell_3 M^4}{F^2} + \frac{M^2}{2F^2}\,\frac{1}{i}\Delta(0, M^2) + O(M^6)$$

6. Start from the symmetry property of the effective action,

$$S_{\mathsf{QCD}}\{v', a', s', p', \theta'\} = S_{\mathsf{QCD}}\{v, a, s, p, \theta\} - \int dx \langle \beta \Omega \rangle,$$

and show that this relation in particular implies the Ward identity

$$
\begin{aligned}
\partial_\mu^x \langle 0|\, T A_a^\mu(x)\, P_b(y)\, |0\rangle = &-\tfrac{1}{4}\, i\, \delta(x-y) \langle 0|\, \overline{q}\{\lambda_a, \lambda_b\} q\, |0\rangle \\
&+ \langle 0|\, T\overline{q}(x)\, i\gamma_5 \{m, \tfrac{1}{2}\lambda_a\} q(x)\, P_b(y)\, |0\rangle
\end{aligned}
$$

$a = 1, \ldots, 8,\ b = 0, \ldots, 8$

7. What is the Ward identity obeyed by the singlet axial current,

$$\partial_\mu^x \langle 0|\, T A_0^\mu(x)\, P_b(y)\, |0\rangle = \ ?$$

19. Expansion in powers of m_s

- The χPT formulae for the expansion of many quantities of physical interest in powers of m_u, m_d, m_s have been worked out to NNLO, not only masses and decay constants, also form factors, $\eta \to 3\pi$, ...
- Theoretical reasoning:
- Pion physics: expansion in powers of m_u, m_d works very well.
- Physics of the strange particles: SU(3)$_V$ is an approximate symmetry.

  $\Rightarrow$ Symmetry breaking parameter $m_s - m_{ud}$ must be small, meaningful to expand in powers of $m_s - m_{ud}$.
- Since $m_u, m_d \ll m_s$

  $\Rightarrow$ m_s can be treated as a perturbation

  $\Rightarrow$ Expect expansion in powers of m_s to work, but convergence to be comparatively slow
- I do not know of an alternative explanation of the empirical fact that SU(3) is an approximate symmetry.

19.1 Form of the effective Lagrangian

- If all three light quark masses vanish, the QCD Lagrangian is invariant under SU(3)$_R$×SU(3)$_L$
- The spontaneous breakdown to SU(3)$_V$ generates eight Nambu-Goldstone bosons:

  $\pi^\pm,\ \pi^0,\ K^\pm,\ K^0,\ \bar{K}^0,\ \eta$

⇒ Effective fields can be collected in $U \in$ SU(3)

$U(x) = \exp i\,\pi(x)$

$\pi(x) = \lambda_1 \pi^1(x) + \ldots + \lambda_8 \pi^8(x)$

- Symmetry again fixes the leading term in $\mathcal{L}_{eff}$:

$$\boxed{\mathcal{L}^{(2)} = \frac{F_0^2}{4}\,\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U \chi^\dagger \rangle + \frac{H_0}{12} D_\mu\theta D^\mu\theta}$$

$$\chi \equiv 2\,B_0\,(s + ip)\,, \qquad \langle X\rangle \equiv \mathrm{tr}(X)$$

- U is now 3x3, χ is 3x3, otherwise the form of the Lagrangian is the same as for SU(2)$_R$×SU(2)$_L$:
- Symmetry does not determine F_0, B_0, H_0

$$\boxed{\mathcal{L}^{(2)} = \frac{F_0^2}{4}\langle D_\mu U D^\mu U^\dagger + \chi U^\dagger + U\chi^\dagger\rangle + \frac{H_0}{12} D_\mu\theta D^\mu\theta}$$

$$\chi \equiv 2\,B_0\,(s + ip)\,, \qquad \langle X\rangle \equiv \mathrm{tr}(X)$$

- Significance of F_0, B_0: leading terms in the expansion of the decay constants and meson masses in powers of the quark masses:

$$\langle 0|\bar{u}\gamma_\mu\gamma_5 d|\pi^-\rangle = \sqrt{2}F_0\,p^\mu\{1 + O(m)\}$$
$$\langle 0|\bar{u}\gamma_\mu\gamma_5 s|K^-\rangle = \sqrt{2}F_0\,p^\mu\{1 + O(m)\}$$
$$\langle 0|\bar{d}\gamma_\mu\gamma_5 s|K^0\rangle = \sqrt{2}F_0\,p^\mu\{1 + O(m)\}$$

$$M_{\pi^-}^2 = (m_u + m_d)B_0 + O(m^2)$$
$$M_{K^-}^2 = (m_u + m_s)B_0 + O(m^2)$$
$$M_{K^0}^2 = (m_d + m_s)B_0 + O(m^2)$$

- The expansion includes m_s

$\Rightarrow$ F_0, B_0 are independent of m_u, m_d, m_s

- In the effective theory built on $\mathrm{SU(2)_R}\times\mathrm{SU(2)_L}$ m_s is not an expansion parameter

$\Rightarrow$ F, B do depend on m_s

$$F_0 = F|_{m_s\to 0} \qquad\qquad B_0 = B|_{m_s\to 0}$$

- Next-to-leading order:

$$\begin{aligned}\mathcal{L}^{(4)} = & L_1\langle D_\mu U D^\mu U^\dagger\rangle^2 + L_2\langle D_\mu U D_\nu U^\dagger\rangle\langle D^\mu U D\nu U^\dagger\rangle \\ & + L_3\langle D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\rangle + L_4\langle D_\mu U D^\mu U^\dagger\rangle\langle \chi U^\dagger + U\chi^\dagger\rangle \\ & + L_5\langle D_\mu U D^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger)\rangle + L_6\langle \chi U^\dagger + U\chi^\dagger\rangle^2 \\ & + L_7\langle \chi U^\dagger - U\chi^\dagger\rangle^2 + L_8\langle \chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger\rangle \\ & - iL_9\langle F^{\mathsf{R}}_{\mu\nu} D^\mu U D^\nu U^\dagger + F^{\mathsf{L}}_{\mu\nu} D^\mu U^\dagger D^\nu U\rangle + L_{10}\langle F^{\mathsf{R}}_{\mu\nu} U F^{\mu\nu \mathsf{L}} U^\dagger\rangle \\ & + H_1\langle F^{\mathsf{R}}_{\mu\nu} F^{\mu\nu\mathsf{R}} + F^{\mathsf{L}}_{\mu\nu} F^{\mu\nu\mathsf{L}}\rangle + H_2\langle \chi\chi^\dagger\rangle\end{aligned}$$

10 LEC + 2 CT: $L_1, \ldots, L_{10}$; H_1, H_2

$\Rightarrow$ $m_u, m_d, m_s, F_0, B_0, L_4, L_5, L_6, L_7, L_8$ determine

$M_{\pi^\pm}, M_{\pi^0}, M_{K^\pm}, M_{K^0}, M_{\bar{K}^0}, M_\eta, F_{\pi^\pm}, F_{\pi^0}, F_{K^\pm}, F_{K^0}, F_{\bar{K}^0}, F_\eta$ to NLO

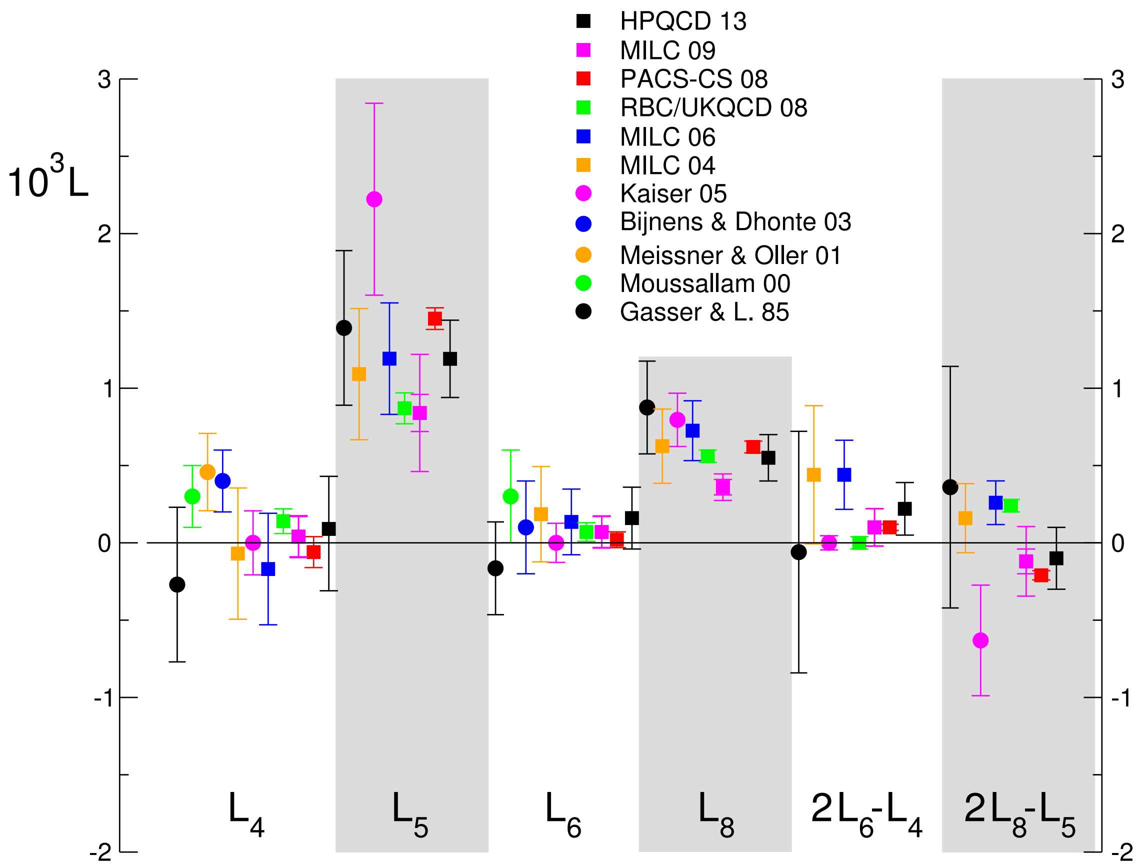
- compare SU(2)xSU(2):

7 LEC + 3 CT: $\ell_1, \ldots, \ell_7$; h_1, h_2, h_3

$\Rightarrow$ $m_u, m_d, F, B, \ell_3, \ell_4, \ell_7$ determine

$M_{\pi^\pm}, M_{\pi^0}, F_{\pi^\pm}, F_{\pi^0}$ to NLO

19.2 Results for L_4, L_5, L_6, L_8



Numerical values shown refer to running scale $\mu = M_\rho$

⇒ For PACS-CS, only the statistical errors are indicated

- The crude estimates given in 1985 for the LEC relevant at NLO are confirmed
- However: not all of the lattice data on the quark mass dependence of M_π, M_K, F_π, F_K are well described by the χPT formulae

- m_s is not very small, terms of order m_s^2 yield sizable corrections.
- Often, m_s is taken in the vicinity of the physical value while m_{ud} is significantly larger than the physical value

$\Rightarrow$ M_K, M_η are larger than the physical values, may be beyond reach

- The constants relevant at NNLO are still poorly known. Often, theoretical estimates are used, obtained by saturating sum rules with resonance contributions. Those constants that govern the dependence on the quark masses, however, represent integrals over scalar spectral functions. Scalar meson dominance does not work!

$\Rightarrow$ Theoretical estimates can at best indicate the order of magnitude.

- The lattice approach is the ideal method for the determination of the LEC !
- Please do not use 'theory' for the LEC.

20. Zweig rule

- Concerns the role played by the sea quarks in physical matrix elements.

Okubo 1963, Zweig 1964, Iizuka 1966

- Leading low energy constants in the effective Lagrangian of SU(2)×SU(2): F and B

$$\{F, B, \Sigma\} = \left\{F_\pi, \frac{M_\pi^2}{m_u + m_d}, |\langle 0|\bar{u}u\,|0\rangle|\right\}_{m_u, m_d \to 0}$$

- Low energy theorem: $\Sigma = F^2 B$
exact, holds for any value of m_s.
- Zweig rule: F and B are independent of m_s.
- F_0, B_0, Σ_0: values for $m_s = 0$
- *Paramagnetic inequalities*: both F and Σ decrease if m_s is taken smaller

$F > F_0\,,\ \Sigma > \Sigma_0$ Descotes-Genon, Girlanda & Stern 2000

- Behaviour if N_c becomes large:

  F, B, Σ become independent of m_s if $N_c \to \infty$

  $F/F_0 \to 1$, $B/B_0 \to 1$, $\Sigma/\Sigma_0 \to 1$

$\Rightarrow$ The differences F/F_0-1, B/B_0-1, Σ/Σ_0-1 measure the violations of the Zweig rule

- Expansion to NLO involves the low energy constants L_4 and L_6 of the SU(3)×SU(3) Lagrangian:

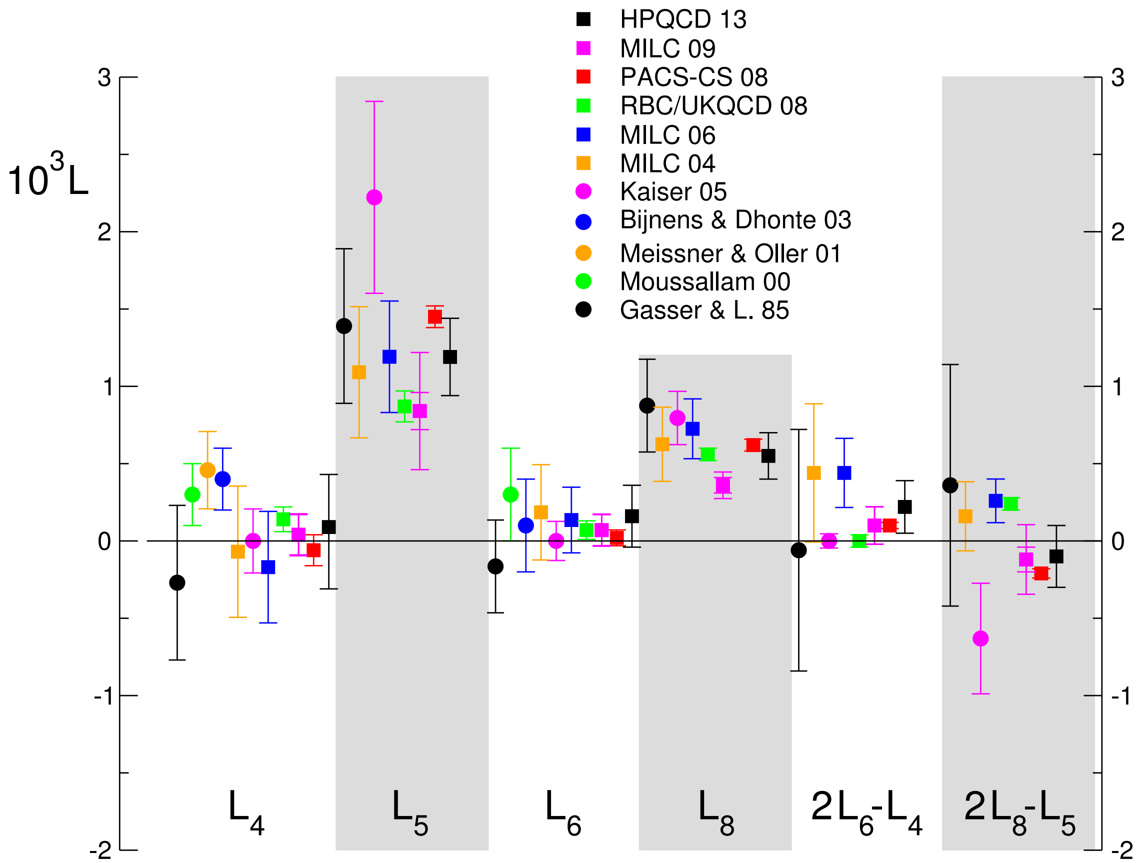
$$F/F_0 = 1 + \frac{8\,\overline{M}_K^2}{F_0^2}\,L_4 + \chi\text{log} + \ldots$$

$$\Sigma/\Sigma_0 = 1 + \frac{32\,\overline{M}_K^2}{F_0^2}\,L_6 + \chi\text{log} + \ldots$$

$$B/B_0 = 1 + \frac{16\,\overline{M}_K^2}{F_0^2}\,(2L_6 - L_4) + \chi\text{log} + \ldots$$

$$\overline{M}_K^2 \equiv m_s\,B_0$$

$\Rightarrow$ The LEC L_4 and L_6 measure the deviations from the Zweig rule



- The LEC appear to obey the Zweig-rule reasonably well: the values for L_4, L_6, $2L_6 - L_4$ are consistent with zero
- Lattice data leave much to be desired: only two papers without red tags in FLAG review: MILC (2009), HPQCD (2013).

- Inserting the lattice results for L_4, L_6 in the NLO formulae of χPT, I get

	F/F_0	B/B_0	Σ/Σ_0
MILC (2009)	1.12(4)	1.10(7)	1.34(13)
HPQCD (2013)	1.10(8)	1.12(8)	1.32(28)
GL (1985)	1.0(1)	1.0(2)	1.0(3)

$\Rightarrow$ Evidence for small Zweig rule violations, consistent with the crude old estimates.

The Zweig rule violations roughly amount to a common change in scale:

$$F \simeq Z F_0 \quad B \simeq Z B_0 \quad \Rightarrow \quad \Sigma \simeq Z^3 \Sigma_0$$

with $\quad Z \simeq 1.10(5)$

- Paramagnetic inequalities of Descotes-Genon, Girlanda & Stern are confirmed.

- MILC has evaluated the ratios to all orders in m_s:

	F/F_0	B/B_0	Σ/Σ_0
NLO	1.12(4)	1.10(7)	1.34(13)
all orders	1.10(4)	1.20(7)	1.48(16)

For F/F_0, the corrections are small, but for B/B_0, the central values of the terms of order m_s and m_s^2 (or higher) are of the same size . . .

$\Rightarrow$ The Zweig rule deserves more attention !

- HPQCD instead evaluated the quark condensates at the physical quark masses:

$$\frac{\langle 0|\bar{s}s\,|0\rangle}{\langle 0|\bar{u}u\,|0\rangle} = 1.08(16)(1)$$

Confirms that SU(3) is a decent approximate symmetry: the symmetry breaking generated by $m_s - m_{ud}$ is too small to stick out from the noise of the calculation.

21. Quark mass ratios

21.1 Isospin breaking

- The symmetry properties of the vacuum shield the pions from isospin breaking:

  The difference between m_u and m_d only generates a tiny effect of order

  $M^2_{\pi^+} - M^2_{\pi^0} \propto (m_u - m_d)^2$.

⇒ The mass difference between π^0 and π^+ is due almost exclusively to electromagnetism.

⇒ More easy to determine the mean mass $m_{ud} \equiv \frac{1}{2}(m_u + m_d)$ than the difference $m_u - m_d$.

- Estimate the e.m. self-energies with the Dashen theorem:

$$M^2_{K^+}\big|_{e.m.} = M^2_{\pi^+}\big|_{e.m.} \qquad M^2_{\pi^0}\big|_{e.m.} = M^2_{K^0}\big|_{e.m.} = 0$$

21.2 Quark mass ratios at leading order

- Solve the tree level mass formulae for the ratios m_s/m_{ud} and m_u/m_d: Weinberg 1977

$$\frac{m_s}{m_{ud}} = \frac{M^2_{K^+} + M^2_{K^0} - M^2_{\pi^+}}{M^2_{\pi^0}} = 25.9$$

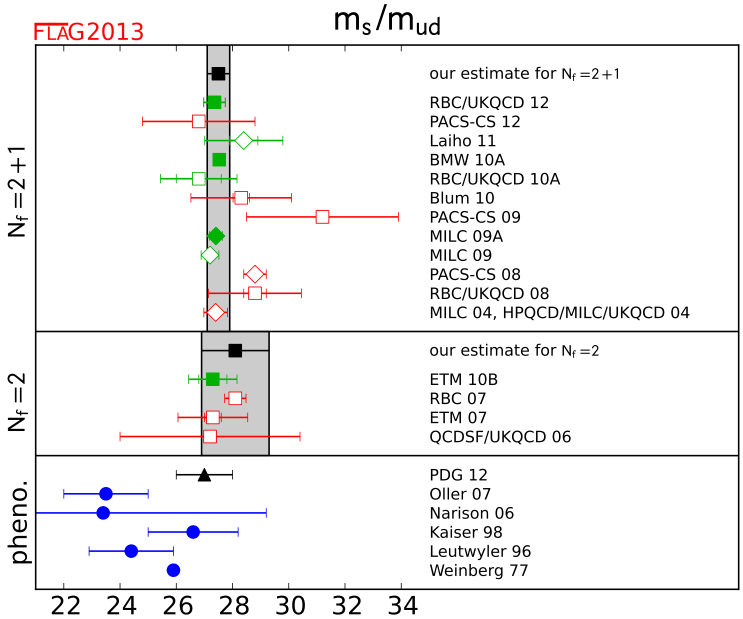
$$\frac{m_u}{m_d} = \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}} = 0.56$$

- Low energy theorems, valid to leading order of the chiral expansion.

  Corrections from higher orders ? Could they strongly modify the numerical result ?

  What is the uncertainty to be attached to these predictions ?

21.3 Lattice results for m_s/m_{ud}



lattice average quoted in FLAG 2013:

$$\frac{m_s}{m_{ud}} = 27.46(15)(41)$$

$$27.46 = 25.9 + 1.6$$

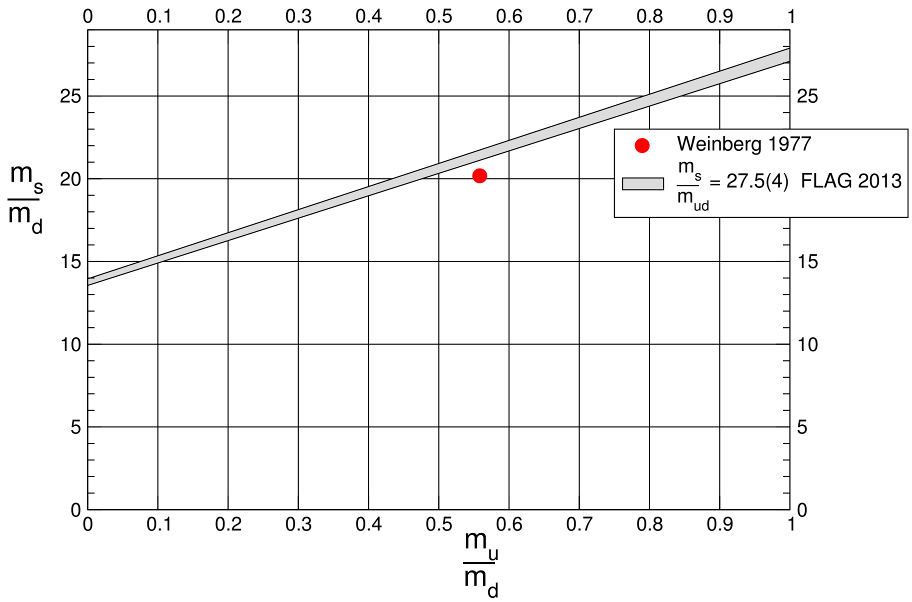
$\Uparrow$ $\Uparrow$

leading order higher orders

$\Rightarrow$ correction is small, leading term of chiral perturbation series dominates

accuracy reached: 1.6 %

21.4 m_s/m_d versus m_u/m_d



- Most lattice calculations are done in pure QCD.
- For m_s/m_{ud}, this is a good approxiation, because the uncertainties in the violations of the Dashen theorem do not strongly affect this ratio.
- For m_u/m_d, the situation is different. Lattice simulations of QCD + QED cannot be done with the same level of confidence as for QCD alone: not all systematic errors are under control (quenched photons, finite size effects for interactions of long range).

21.5 Low energy theorem valid to NLO

- The lattice result for m_s/m_{ud} determines the size of the correction in the relation

$$\frac{M_K^2}{M_\pi^2} = \frac{m_s + m_{ud}}{m_u + m_d}\left\{1 + \Delta_M\right\}$$

$m_s/m_{ud} = 27.5 \pm 0.4 \Rightarrow \Delta_M = -0.057 \pm 0.013.$

- Remarkably, chiral symmetry implies that the correction of NLO in the ratio of mass splittings is the same:

$$\frac{M_{K^0}^2 - M_{K^+}^2}{M_K^2 - M_\pi^2} = \frac{m_d - m_u}{m_s - m_{ud}}\left\{1 + \Delta_M + O(\mathcal{M}^2)\right\}$$

Hence the quark mass ratio

$$Q^2 \equiv \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2}$$

is given by a ratio of meson masses, up to corrections of NNLO:

$$Q^2 = \frac{M_K^2 - M_\pi^2}{M_{K^0}^2 - M_{K^+}^2} \cdot \frac{M_K^2}{M_\pi^2}\left\{1 + O(\mathcal{M}^2)\right\}$$

Gasser & L. 1985

21.6 Consequences for Q

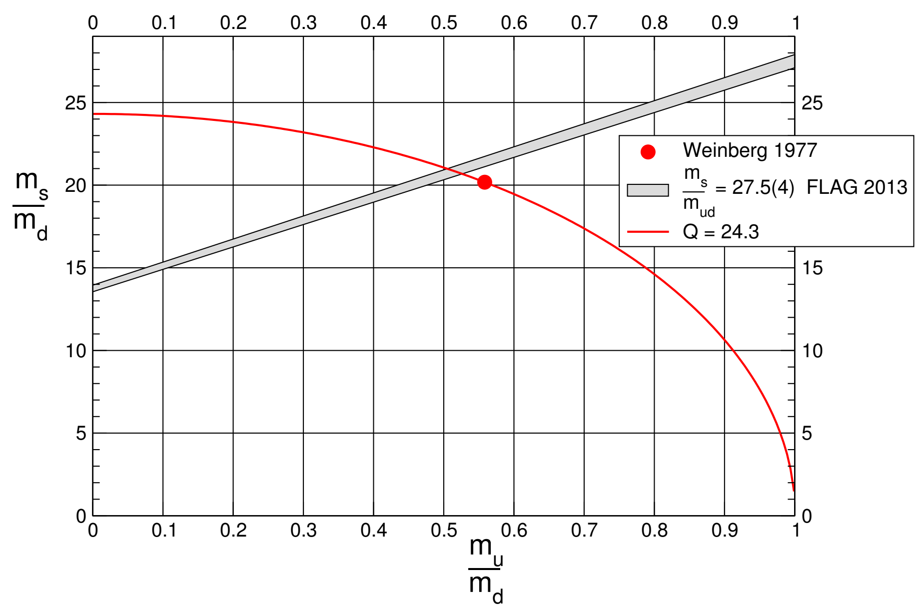
- Insert Weinberg's leading order ratios

$\Rightarrow$ $Q = 24.3.$

- Q^2 is a ratio of quark mass squares

$\Rightarrow$ a given value of Q imposes a homogeneous quadratic constraint on m_u, m_d, m_s

$\Rightarrow$ represents an ellipse in the plane of the quark mass ratios:



- Critical input here is the "Dashen theorem": Weinberg's estimates for the quark mass ratios account for QED only to LO.

22. The decay $\eta \to 3\pi$

- The decay $\eta \to 3\pi$ provides a better handle on Q than the mass splitting between K^+ and K^0, because the e.m. interaction is suppressed (Sutherland's theorem).

- For $e = 0$ and $m_u = m_d$, isospin is conserved, hence G-parity is conserved.

  In this limit, the η is a stable particle: $G_\eta = 1$, $G_\pi = -1$.

⇒ Since the e.m. contributions are tiny, the transition amplitude is to a very good approximation proportional to $(m_u - m_d)$.

22.1 Tree level

- Parameter free prediction for the leading term of the chiral perturbation series:

$$A(\eta \to \pi^+\pi^-\pi^0) = -\frac{\sqrt{3}}{4} \cdot \frac{m_d - m_u}{m_s - m_{ud}} \cdot \frac{s - \frac{4}{3}M_\pi^2}{F_\pi^2}$$

- Compare leading term in the chiral expansion of the $\pi\pi$ scattering amplitude:

$$A(\pi\pi \to \pi\pi) = \frac{s - M_\pi^2}{F_\pi^2}$$

- In both cases, the leading term is linear in s and contains an Adler zero

$\pi\pi$ scattering	η decay
$s_A = M_\pi^2$	$s_A = \frac{4}{3}M_\pi^2$

- The analytic structure of the two amplitudes is very similar.

- In both cases, the higher order contributions of the chiral perturbation series are dominated by the final state interaction among the pions.

22.2 One loop

- Most remarkable property of the one loop representation: expressed in terms of F_π, F_K, M_π, M_K, M_η, Q, all LEC except L_3 drop out.

Gasser & L. 1985

$$A(\eta \to \pi^+\pi^-\pi^0) = -\frac{1}{Q^2} \cdot \frac{M_K^2(M_K^2 - M_\pi^2)}{3\sqrt{3}M_\pi^2 F_\pi^2} \cdot M(s,t,u)$$

- Moreover, L_3 concerns the momentum dependence of the amplitude, can be determined quite well from $\pi\pi$ scattering.

⇒ At one loop, the result for the rate is of the form

$$\Gamma_{\eta\to\pi^+\pi^-\pi^0} = \frac{C}{Q^4} \qquad Q^2 \equiv \frac{m_s^2 - m_{ud}^2}{m_d^2 - m_u^2}$$

where C is a known constant ⇒ Q can be determined from the observed rate.

- The main problem is not the uncertainty in L_3, but the contributions from higher orders.

- In 1985, we estimated the uncertainty in the result for Q at

$$\frac{1}{Q^2} = (1.9 \pm 0.3) \cdot 10^{-3} \quad \leftrightarrow \quad Q = 22.9^{+2.1}_{-1.6}$$

Gasser & L. 1985

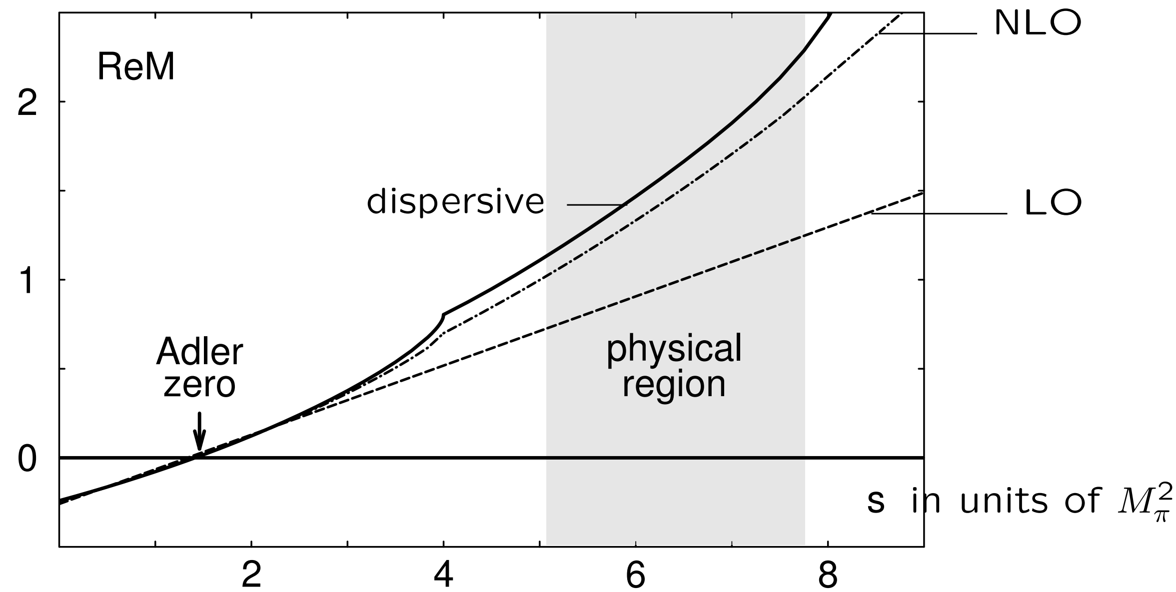
- The result is consistent with the value $Q = 24.3$ obtained from the kaon mass difference with the Dashen theorem, but the uncertainties are large.

22.3 Dispersion theory

- The structure of the decay amplitude is governed by the final state interaction.

  Standard method for the analysis of this interaction: dispersion theory.

- Main difference to $\pi\pi$ scattering: the subtraction constants relevant for $\eta \to 3\pi$ cannot be predicted to the same precision.
- Can analyze $\pi\pi$ scattering by treating only m_u and m_d as small: SU(2)$\times$SU(2)
- In η decay, need to treat m_s as an expansion parameter as well: SU(3)$\times$SU(3)
- Only the occurrence of an Adler zero follows from SU(2)$\times$SU(2) symmetry alone.
- The subtraction constants can be estimated by comparing the dispersive and chiral representations at small values of s, t or u and requiring the occurrence of an Adler zero at the proper place.



Anisovich & L. 1996

⇒ Final state interaction amplifies the transition.

- Effect of the higher order contributions on the result for Q is modest:

$Q = 22.4 \pm 0.9$ Kambor, Wiesendanger & Wyler 1996

$Q = 22.7 \pm 0.8$ Anisovich & L. 1996

- Confirmed the one loop result, $Q = 22.9^{+2.1}_{-1.6}$, uncertainty reduced by a factor of 2.

22.4 Recent work on $\eta \to 3\pi$

- In the meantime, the experimental situation improved a lot: KLOE, MAMI, WASA.
- At low energies, the $\pi\pi$ phase shifts are now known to remarkable accuray:
  - Low energy precision experiments (E865, NA48, DIRAC).
  - Low energy theorems for scattering lengths.
  - Dispersion theory (Roy equations).
- χPT has been worked out to NNLO.

  Bijnens & Ghorbani 2007
- At the precision reached, isospin breaking needs to be accounted for. Ditsche, Kubis & Meissner 2009
- Nonrelativistic effective theory.

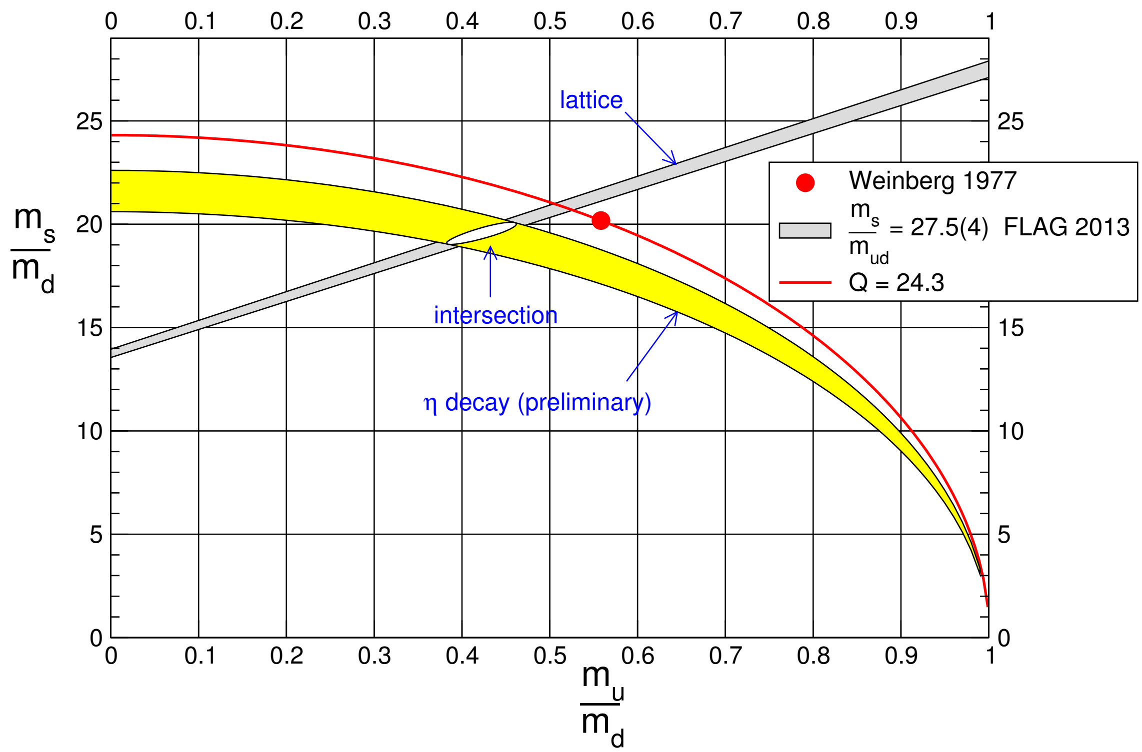
  Gullström, Kupsc & Rusetsky 2009

  Schneider, Kubis & Ditsche 2011

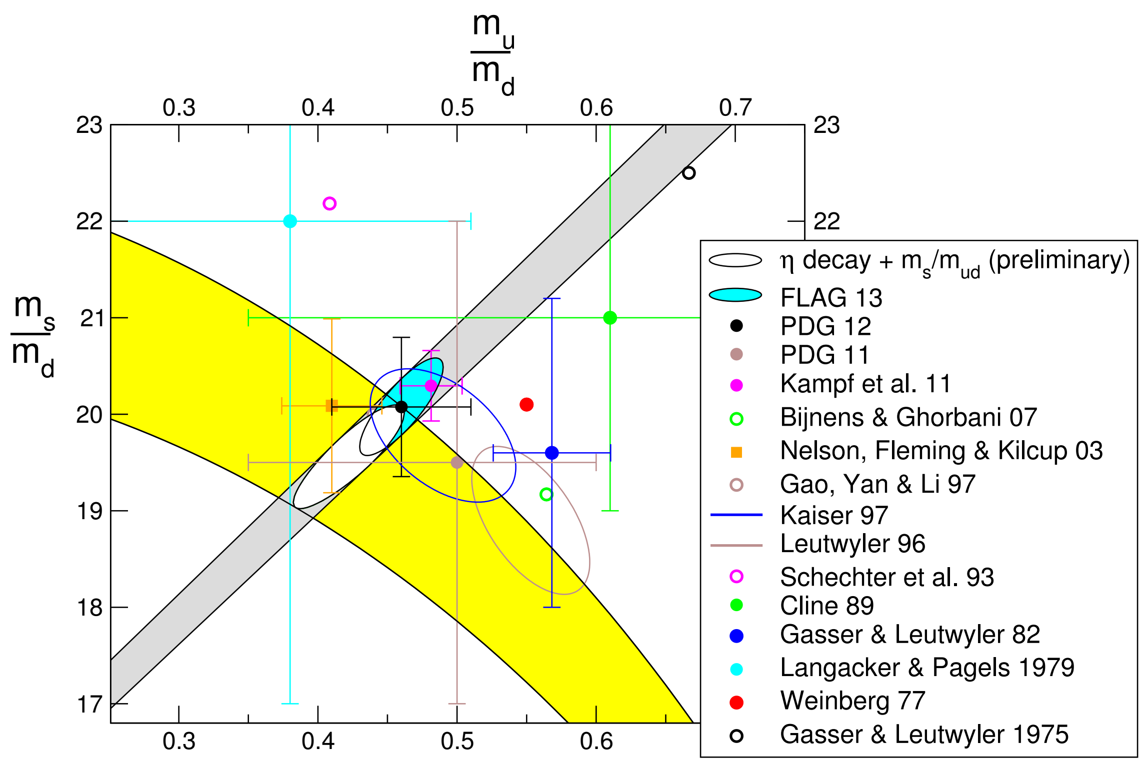
- Improved dispersive analysis is under way

Colangelo, Lanz, L. & Passemar

- Preliminary results for the quark mass ratios



- Intersection moves to values of m_u/m_d and m_s/m_d that are somewhat smaller than those obtained with the LO mass formulae of Weinberg.



- The preliminary results for the ratio m_u/m_d are consistent with the lattice averages quoted by FLAG, but tend to be somewhat smaller.

22. Conclusions for SU(3)×SU(3)

- Expansion in powers of m_s appears to work:

  In all cases I know, where the calculation is under control, the truncation at low order yields a decent approximation

⇒ The picture looks coherent, also for SU(3)×SU(3)

- $m_s \gg m_u, m_d \Rightarrow$ higher orders more important
- For many observables ∃ representation to NNLO

  Bijnens and collaborators

- Main problem: new LEC relevant at NNLO

- ∃ estimates based on resonance models
- Vector meson dominance √
- Dependence on m_u, m_d, m_s: scalar resonances
- Scalar meson dominance ?

- Lattice results now start providing more precise values for the LEC, but the settling of dust is a slow process . . .

23. QCD at nonzero temperature

- Most likely distribution of a given energy: thermal equilibrium, characterized by T
- Partition function: $Z = \mathrm{Tr}\, e^{-\frac{H}{T}}$

23.1 Magnets

$$H = H_0 - \int d^3x\, \vec{H} \cdot \vec{M}$$

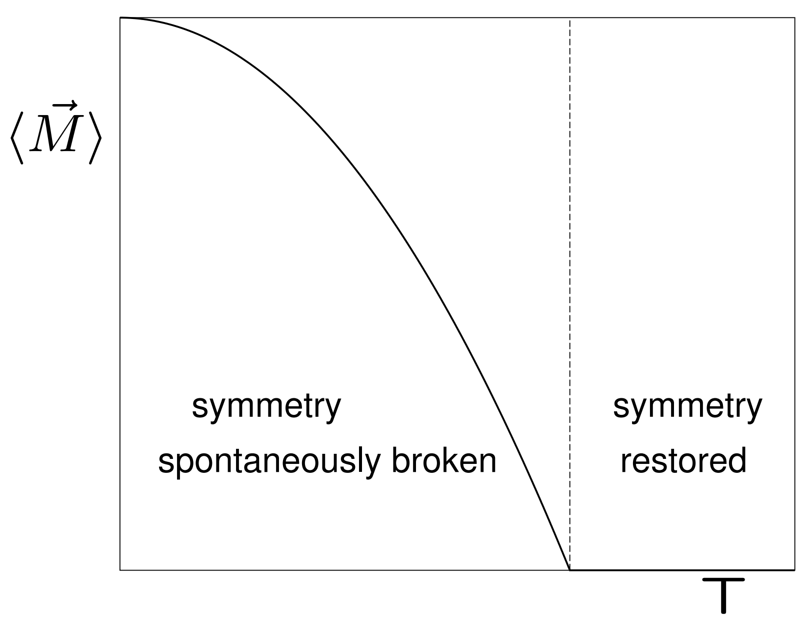
$\vec{H}$: external magnetic field
$\vec{M}$: magnetization

- Expectation value of magnetization:

$$\langle \vec{M} \rangle = \frac{1}{Z}\, \mathrm{Tr} \left\{ e^{-\frac{H}{T}} \vec{M} \right\}$$

- $\langle \vec{M} \rangle$ is parallel to $\vec{H}$
- *Spontaneous magnetization:*

  $\langle \vec{M} \rangle$ stays $\neq 0$ if $\vec{H} \to 0$



Temperature dependence of spontaneous magnetization

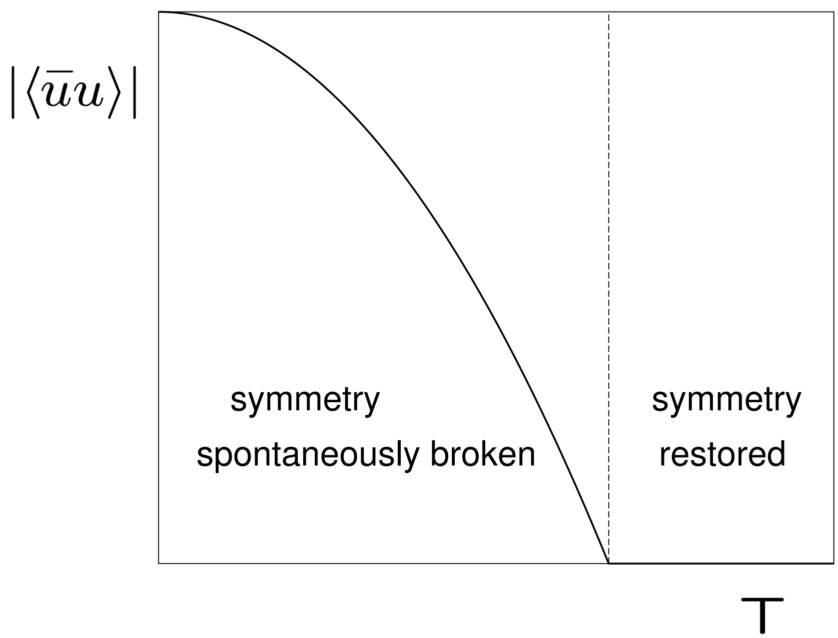
23.2 QCD

$$H = H_0 + \int d^3x \{ m_u \bar{u}u + m_d \bar{d}d \}$$

- Partition function: $Z = \mathrm{Tr}\, e^{-\frac{H}{T}}$
- Quark condensate at nonzero temperature:

$$\langle \bar{u}u \rangle = \frac{1}{Z} \mathrm{Tr} \left\{ e^{-\frac{H}{T}} \, \bar{u}u \right\}$$

- For $T \to 0$, $\langle \bar{u}u \rangle$ tends to $\langle 0 | \bar{u}u \, | 0 \rangle$



Temperature dependence of quark condensate

- Symmetry relevant here: $\mathrm{SU(2)_R \times SU(2)_L}$

  Symmetry is exact only for $m_u, m_d \to 0$

23.3 Partition function of QCD at low T

- Insert complete set of states $H|n\rangle = E_n|n\rangle$

  $Z = \text{Tr}\, e^{-\frac{H}{T}} = \sum_n e^{-\frac{E_n}{T}}$

- Only states with $E_n \lesssim T$ contribute

  At T = 0 only the vacuum survives, $|n\rangle = |0\rangle$

- At low T, the next most important contribution stems from the pions

- Pions of low energy behave like free particles, only interact weakly

⇒ At low energies, the partition function of QCD describes a gas of free pions

23.4 Partition function of free Bose gas

Bose 1924, Einstein 1925

- Complete set of one particle modes
- Label the modes with $k = 1, 2, 3, \ldots$
- Example: box of size $L \times L \times L$, plane waves

  $\vec{p} = \frac{2\pi}{L}\{k_1, k_2, k_3\}$, $k_r \in Z$

  $k \leftrightarrow \{k_1, k_2, k_3\}$

- bosons: n_k = 0,1,2,... particles in each mode
  fermions: n_k = 0 or 1

$\Rightarrow$ Complete set of states for the entire gas:
$|n\rangle = |n_1, n_2, \ldots\rangle$

- Energy in mode k: $\omega_k = \sqrt{m^2 + \vec{p}^2}$
- Energy of the gas in such a state

$E_n = E_0 + n_1\omega_1 + n_2\omega_2 + \ldots$
$\Uparrow$ vacuum energy

$$Z = \sum_{n_1, n_2, \ldots} e^{-\frac{1}{T}\{E_0 + n_1\omega_1 + n_2\omega_2 + \ldots\}}$$

$$= \sum_{n_1, n_2, \ldots} e^{-\frac{E_0}{T}} \times e^{-\frac{n_1\omega_1}{T}} \times e^{-\frac{n_2\omega_2}{T}} \ldots$$

$$= e^{-\frac{E_0}{T}} \times \frac{1}{1 - e^{-\frac{\omega_1}{T}}} \times \frac{1}{1 - e^{-\frac{\omega_2}{T}}} \ldots$$

$$= e^{-\frac{E_0}{T}} \times \prod_k \frac{1}{1 - e^{-\frac{\omega_k}{T}}}$$

$$\ell n\, Z = -\frac{E_0}{T} - \sum_k \ell n(1 - e^{-\frac{\omega_k}{T}})$$

- Number of states in $\Delta^3 p$: $\quad \frac{\Delta^3 p}{(2\pi/L)^3} = \frac{\Delta^3 p V}{(2\pi)^3}$

  This reproduces a general rule of statistical mechanics: the volume element $\Delta^3 p \Delta^3 x$ of phase space contains $\Delta^3 p \Delta^3 x / h^3$ quantum states

$$\Rightarrow \ell n\, Z = -\frac{E_0}{T} - \frac{V}{(2\pi)^3} \int d^3 p\, \ell n(1 - e^{-\frac{\omega_{\vec{p}}}{T}})$$

23.5 Melting of the condensate

- QCD with two light flavours: 3 NGBs

$$\ell n\, Z_{\mathrm{QCD}} = -\frac{E_0}{T} - 3\frac{V}{(2\pi)^3}\int d^3p\, \ell n(1 - e^{-\frac{\omega_{\vec{p}}}{T}}) + \ldots$$

- Calculate the condensate from the partition function

$$\frac{\partial Z_{\mathrm{QCD}}}{\partial m_u} = \frac{\partial\, \mathrm{Tr}\, e^{-\frac{H}{T}}}{\partial m_u} = -\frac{1}{T}\mathrm{Tr}\left\{e^{-\frac{H}{T}}\frac{\partial H}{\partial m_u}\right\}$$

$$H = H_0 + \int d^3x\{m_u \bar{u}u + m_d \bar{d}d\}$$

$$\frac{\partial H}{\partial m_u} = \int d^3x\, \bar{u}u$$

$$\frac{\partial Z_{\mathrm{QCD}}}{\partial m_u} = -\frac{1}{T}\int d^3x \mathrm{Tr}\left\{e^{-\frac{H}{T}}\bar{u}u\right\} = -\frac{V}{T}\mathrm{Tr}\left\{e^{-\frac{H}{T}}\bar{u}u\right\}$$

$$= -\frac{V}{T}\langle\bar{u}u\rangle Z_{\mathrm{QCD}}$$

$$\Rightarrow \boxed{\langle\bar{u}u\rangle = -\frac{T}{V}\frac{\partial\, \ell n\, Z_{\mathrm{QCD}}}{\partial m_u}}$$

$$\ell n\, Z_{\text{QCD}} = -\frac{E_0}{T} - 3\frac{V}{(2\pi)^3}\int d^3p\, \ell n(1 - e^{-\frac{\omega_{\vec{p}}}{T}})$$

- First term dominates at low T:

$$\langle \bar{u}u \rangle = \frac{1}{V}\frac{\partial E_0}{\partial m_u} + \dots \text{ independent of T}$$

$$\frac{1}{V}\frac{\partial E_0}{\partial m_u} = \langle 0|\, \bar{u}u\, |0\rangle$$

- Second term also depends on m_u, via

$$\omega_{\vec{p}} = \sqrt{M_\pi^2 + \vec{p}^{\,2}} \qquad M_\pi^2 = (m_u + m_d)B + \dots$$

$$\Rightarrow \frac{\partial \omega_{\vec{p}}}{\partial m_u} = \frac{B}{2\omega_{\vec{p}}} = -\frac{\langle 0|\, \bar{u}u\, |0\rangle}{2\omega_{\vec{p}} F^2}$$

$$\boxed{\langle \bar{u}u \rangle = \langle 0|\, \bar{u}u\, |0\rangle \left\{ 1 - \frac{3}{16\pi^3 F^2}\int \frac{d^3p}{\omega_{\vec{p}}}\frac{1}{\left(e^{\frac{\omega_{\vec{p}}}{T}} - 1\right)} + \dots \right\}}$$

$\Rightarrow \langle \bar{u}u \rangle < \langle 0|\, \bar{u}u\, |0\rangle$ quark condensate melts

- Melting for massless quarks:

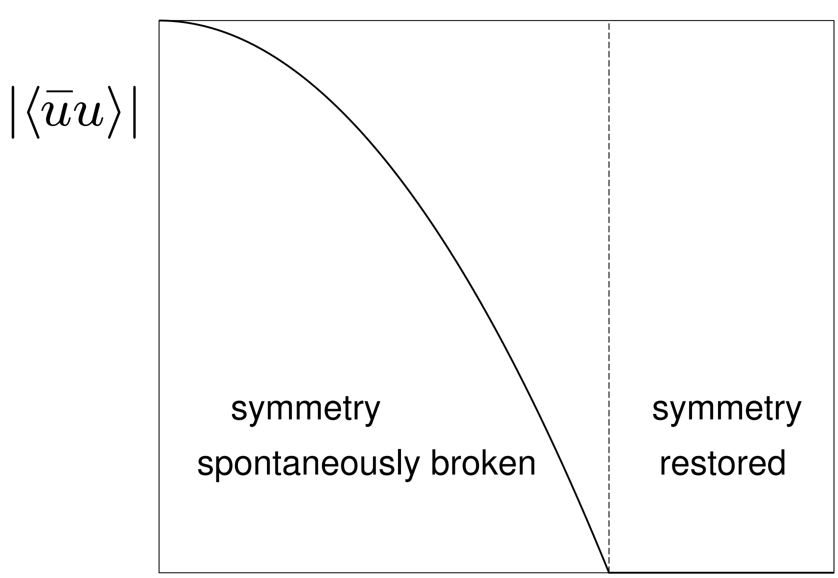
$$m_u, m_d \to 0 \quad \Rightarrow \quad M_\pi = 0 \quad \Rightarrow \quad \omega_{\vec{p}} = |\vec{p}|$$

$$\int \frac{d^3p}{|\vec{p}|} \frac{1}{\left(e^{\frac{|\vec{p}|}{T}} - 1\right)} = \frac{2\pi^3 T^2}{3}$$

$$\Rightarrow \qquad \boxed{\langle \bar{u}u \rangle = \langle 0 | \bar{u}u \, | 0 \rangle \left\{ 1 - \frac{T^2}{8F^2} + O(T^4) \right\}}$$

First two terms in temperature expansion of the quark condensate for $m_u = m_d = 0$. The constant F is the value of F_π in this limit.

Binétruy & Gaillard 1985



23.6 Pressure, energy density

- Free energy: $Z_{\text{QCD}} = e^{-\frac{F_{\text{QCD}}}{T}}$

$$\boxed{F_{\text{QCD}} = E_0 + \frac{3VT}{(2\pi)^3}\int d^3p\,\ell n(1-e^{-\frac{\omega_{\vec{p}}}{T}}) + \ldots}$$

Free energy of noninteracting Bose gas
3 flavours: π^-, π^0, π^+

- Massless pions: $\int d^3p\,\ell n(1-e^{-\frac{|\vec{p}|}{T}}) = -\frac{4\pi^5}{45}T^3$

$\Rightarrow$ $F_{\text{QCD}} = E_0 - \frac{\pi^2}{30}VT^4 + O(T^6)$

- $P_{\text{QCD}} = -\frac{\partial F_{\text{QCD}}}{\partial V} = \frac{\pi^2}{30}T^4 + O(T^6)$ pressure

- $s_{\text{QCD}} = \frac{\partial P_{\text{QCD}}}{\partial T} = \frac{2\pi^2}{15}T^3 + O(T^5)$ entropy density

- $u_{\text{QCD}} = Ts - P = \frac{\pi^2}{10}T^4 + O(T^6)$ energy density

$\Rightarrow$ Leading terms in massless QCD are the same as for black body radiation, except for a factor 3/2 (3 independent pion states of a given momentum, 2 independent photon states)

23.7 Comparison: high temperature

- Asymptotic freedom $\Rightarrow$ at high temperature, QCD also represents a gas of free particles: quarks, gluons
- High temperatures are beyond reach of χPT Instead: perturbation theory in powers of α_s
- Gas of free gluons: energy density, pressure, differ from the expressions obtained for a gas of free pions only by the factor $\frac{2\times 8}{3}$
- Quarks are fermions, obey different statistics

- Partition function of free fermions

$$Z = \sum_{n_1, n_2, \ldots} e^{-\frac{E_0}{T}} \times e^{-\frac{n_1 \omega_1}{T}} \times e^{-\frac{n_2 \omega_2}{T}} \ldots$$

$$Z = e^{-\frac{E_0}{T}} \times (1 + e^{-\frac{\omega_1}{T}}) \times (1 + e^{-\frac{\omega_2}{T}}) \ldots$$

$$= e^{-\frac{E_0}{T}} \times \prod_k (1 + e^{-\frac{\omega_k}{T}})$$

$$\ell n\, Z = -\frac{E_0}{T} + \sum_k \ell n(1 + e^{-\frac{\omega_k}{T}})$$

$$\Rightarrow \ell n\, Z = -\frac{E_0}{T} + \frac{V}{(2\pi)^3} \int d^3p\, \ell n(1 + e^{-\frac{\omega_{\vec{p}}}{T}})$$

$$\Rightarrow \boxed{F = E_0 - \frac{VT}{(2\pi)^3} \int d^3p\, \ell n(1 + e^{-\frac{\omega_{\vec{p}}}{T}})}$$

Free energy of noninteracting fermions

$$\omega_{\vec{p}} = \sqrt{m^2 + \vec{p}^2}$$

- Massless quarks: $\omega_{\vec{p}} = |\vec{p}|$

$$\int d^3p\, \ell n(1 + e^{-\frac{|\vec{p}|}{T}}) = \frac{7\pi^5}{90} T^3$$

$$\Rightarrow F_{\text{quarks}} = E_0 - 3 \cdot 2 \cdot 2 \cdot N_f \, \frac{7\pi^2}{720} T^4 V$$

- Net result for energy density of QCD with N_f massless quarks
  - High temperature:

$$u_{\text{QCD}} = \frac{\pi^2}{30} T^4 \left\{ 8 \underset{\substack{\Uparrow \\ \text{gluons}}}{\cdot} 2 + \frac{7}{8} \cdot 3 \cdot 2 \underset{\substack{\Uparrow \\ \text{quarks}}}{\cdot} 2 \cdot N_f \right\} + \ldots$$

  - Low temperature:

$$u_{\text{QCD}} = \frac{\pi^2}{30} T^4 \cdot (\underset{\substack{\Uparrow \\ \text{Nambu-Goldstone Bosons}}}{N_f^2} - 1) + \ldots$$

- Contributions from other particles at low T:

  $\sim e^{-M_\rho/T} \simeq 0.006$ for $\simeq 150 MeV$

- Probability to find a ρ is small, but:

  many statistically independent states

$\Rightarrow$ Already at T = 130 MeV, more energy is stored in $K, \eta, \rho \ldots$ than in the pions

- Return to the quark condensate at low T

  QCD with N_f massless quark flavours

$$\langle \bar{u}u \rangle = \langle 0|\bar{u}u\,|0\rangle \left\{ 1 - \frac{(N_f^2-1)}{12N_f}\frac{T^2}{F^2} + O(T^4) \right\}$$

- Term of order T^4 ? Very tedious to do the calculation by hand, use the washing machine:

23.8 Chiral Perturbation Theory for $T \neq 0$

- Standard procedure:

  - Path integral representation for the transition amplitude in quantum mechanics

$$\langle x'|e^{-itH}|x\rangle = \int [Dx] e^{i\int_0^t dt' L}$$

  Integration extends over all paths $x(t)$ with $x(0) = x$, $x(t) = x'$

  - Path integral representation for the matrix elements of $e^{-\beta H}$: continue analytically in t to $t = -i\beta$

$$\langle x'|e^{-\beta H}|x\rangle = \int [Dx] e^{-\int_0^\beta dt' L^{eucl}}$$

- Partition function in this notation:

$$\text{Tr}\, e^{-\beta H} = \int dx \langle x|e^{-\beta H}|x\rangle \text{ with } \boxed{\beta = \frac{1}{T}}$$

⇒ Set $x' = x$ and integrate over x

$$\boxed{\text{Tr}\, e^{-\beta H} = \int [Dx] e^{-\int_0^\beta dt' L^{eucl}}}$$

Integration over all paths with $x(\beta) = x(0)$

- Formula also holds for path integral representation of QCD:

$$\text{Tr}\, e^{-\beta H} = \mathcal{N} \int [dG]\, e^{-\int_0^\beta dx^4 \int d^3x \mathcal{L}_{\text{G}}^{eucl}} \det D$$

as well as for the effective theory:

$$\boxed{\text{Tr}\, e^{-\beta H} = \mathcal{N}_{\mathit{eff}} \int [dU] e^{-\int_0^\beta dx^4 \int d^3x \mathcal{L}_{\mathit{eff}}^{eucl}\{U,v,a,s,p,\theta\}}}$$

Integration extends over all periodic fields:

$$U(\vec{x}, \beta) = U(\vec{x}, 0)$$

- Vertices in the effective Lagrangian remain the same: LEC and CT are independent of T

Massless quarks

- Consider QCD with $N_f \geq 2$ massless flavours

$\Rightarrow$ $\exists\, N_f^2 - 1$ massless Nambu-Goldstone-bosons

- χPT provides expansion in powers of T:

$$\langle \bar{u}u \rangle = \langle 0 | \bar{u}u \, | 0 \rangle \left\{ 1 - c_1 \frac{T^2}{F_\pi^2} - c_2 \frac{T^4}{F_\pi^4} - \right.$$
$$\left. - c_3 \frac{T^6}{F_\pi^6} \, \ell n \frac{\Lambda_q}{T} + O(T^8) \right\}$$

$$c_1 = \frac{N_f^2 - 1}{12 N_f}$$ Binétruy & Gaillard 1985

$$c_2 = \frac{N_f^2 - 1}{288 N_f^2}$$ Gasser & L. 1987

$$c_3 = \frac{N_f (N_f^2 - 1)}{1728}$$ Gerber & L. 1989

- Result is exact: the condensate of massless QCD admits a Taylor series expansion in T. The first few coefficients are determined by the value of F_π in massless QCD.

- At order T^3, there is a chiral logarithm; the scale thereof is fixed by the LEC of NLO.

D. Partition function of a free gas

$$Z = \text{Tr}\, e^{-\frac{H}{T}}$$

- Insert complete set of states $H|n\rangle = E_n|n\rangle$
  - Complete set of one particle modes
  - Label the modes with $k = 1, 2, 3, \ldots$
  - Example: box of size $L \times L \times L$, plane waves

    $\vec{p} = \frac{2\pi}{L}\{k_1, k_2, k_3\},\ k_r \in Z$

    $k \leftrightarrow \{k_1, k_2, k_3\}$

- bosons: $n_k = 0,1,2,\ldots$ particles in each mode
  fermions: $n_k = 0$ or 1

$\Rightarrow$ Complete set of states for the entire gas:
$|n\rangle = |n_1, n_2, \ldots\rangle$

- Energy in mode k: $\omega_k = \sqrt{m^2 + \vec{p}^2}$
- Energy of the gas in such a state

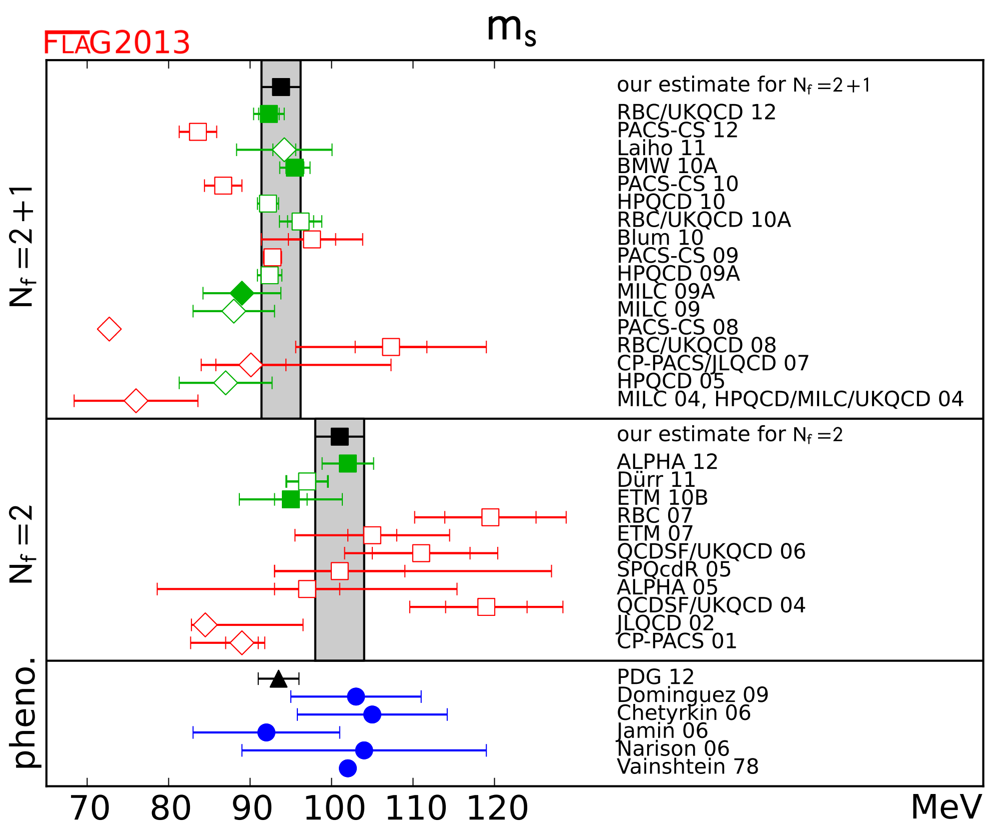
$$E_n = E_0 + n_1\omega_1 + n_2\omega_2 + \ldots$$

$\Uparrow$ vacuum energy

IV. Some recent results

21. Masses of the light quarks

- χPT plays an important role in the analysis of lattice data: describes the dependence of the various observables on the quark masses and on the size of the box in terms of a few LEC



$m_s(2\,\mathrm{GeV}) = 99 \pm 11\,\mathrm{MeV}$ FLAG 2010 (preliminary)

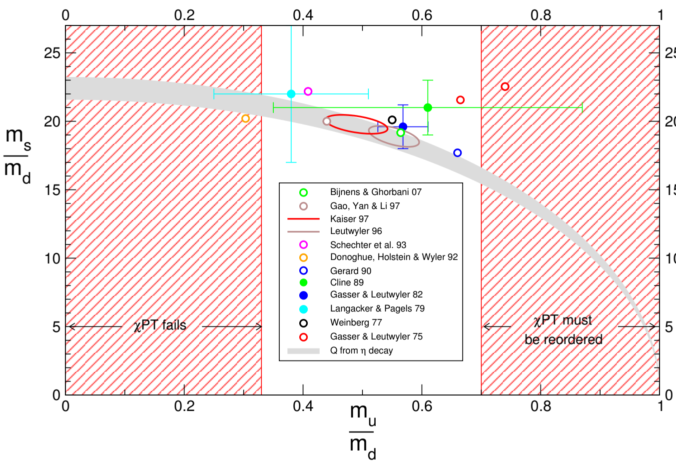
Summary in FLAG 2013:

N_f	m_{ud}	m_s	m_s/m_{ud}
2+1	3.42(6)(7)	93.8(1.5)(1.9)	27.46(15)(41)
2	3.6(2)	101(3)	28.1(1.2)

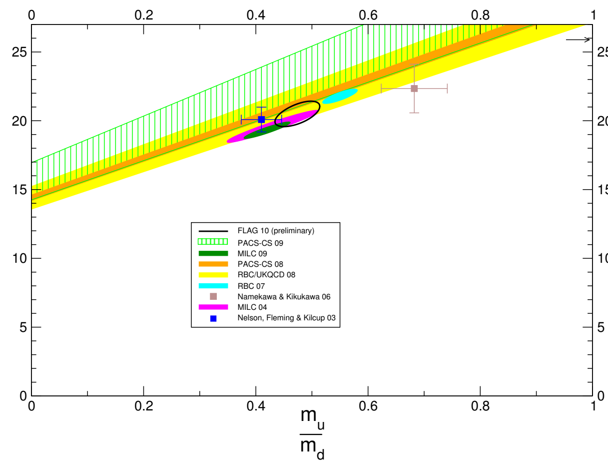
Isospin breaking in the quark masses:

N_f	m_u	m_d	m_u/m_d	Q
2+1	2.16(11)	4.68(14)(7)	0.46(2)(2)	22.6(7)(6)
2	2.40(23)	4.80(23)	0.50(4)	24.3(1.4)(0.6)

Results for quark mass ratios



Phenomenology



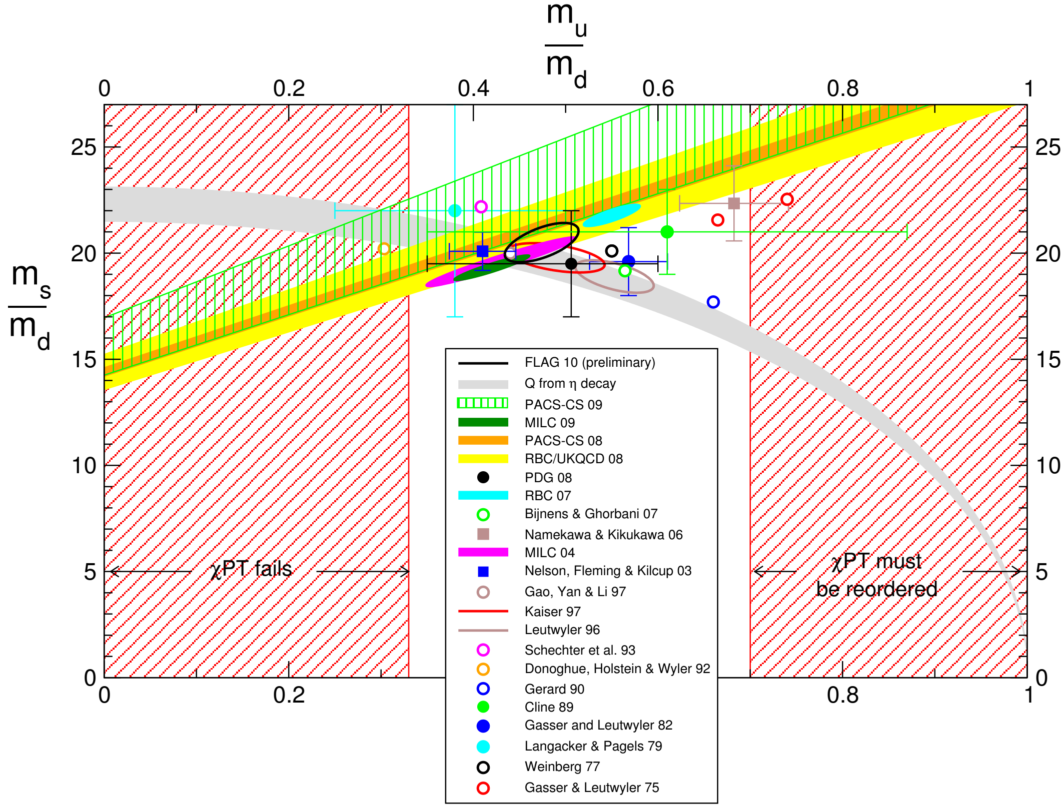
Lattice

$$\frac{m_s}{m_{ud}} = 27.8 \pm 1.0 \qquad \frac{m_u}{m_d} = 0.474 \pm 0.040$$

FLAG 2010 (preliminary)

None of the lattice results is consistent with the "solution" $m_u = 0$ of the strong CP problem

Comparison



mu/md
ms/md
0
0.2
0.4
0.6
0.8
1
25
20
15
10
5
0
χPT fails
χPT must be reordered
FLAG 10 (preliminary)
Q from η decay
PACS-CS 09
MILC 09
PACS-CS 08
RBC/UKQCD 08
PDG 08
RBC 07
Bijnens & Ghorbani 07
Namekawa & Kikukawa 06
MILC 04
Nelson, Fleming & Kilcup 03
Gao, Yan & Li 97
Kaiser 97
Leutwyler 96
Schechter et al. 93
Donoghue, Holstein & Wyler 92
Gerard 90
Cline 89
Gasser and Leutwyler 82
Langacker & Pagels 79
Weinberg 77
Gasser & Leutwyler 75

22. V_{us} and V_{ud}

- Experimental sources for V_{us} and V_{ud}:

superallowed nuclear β transitions	$\lvert V_{ud}\rvert$
$K \to \pi\ell\nu$	$\lvert f_+(0)V_{us}\rvert$
$\pi \to \ell\nu$, $\tau \to \pi\nu$	$\lvert V_{ud}\,F_\pi\rvert$
$K \to \ell\nu$, $\tau \to K\nu$	$\lvert V_{us}\,F_K\rvert$
inclusive τ decays	$\lvert V_{us}\rvert$

- Vector current relevant for nuclear β decay is conserved modulo $m_u - m_d$

$\Rightarrow$ analog of $f_+(0)$ is very close to unity

$$|V_{ud}| = 0.97425 \pm 0.00022$$ Hardy + Towner 2009

- Can determine V_{us} from $K \to \pi\ell\nu$ only if $f_+(0)$ is known. Early determinations were based on χPT prediction for that
- Lattice calculations now provide reliable and precise determination of $f_+(0) \Rightarrow |V_{us}|$
- Results for F_π, F_K do not yet reach sufficient precision, but those for the ratio F_K/F_π do

$\Rightarrow$ $\dfrac{V_{us}}{V_{ud}}$ can be determined from $\dfrac{\Gamma(K \to \ell\nu)}{\Gamma(\pi \to \ell\nu)}$

$\Rightarrow$ can test the Standard Model:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 \stackrel{?}{=} 1$$

$|V_{ub}|$ known well enough, contribution is tiny

- Testing the Standard Model with the lattice data alone

$$|V_u|^2 \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1.002 \pm 0.016$$

- Lattice results for V_{ud} are consistent with the value obtained from nuclear β-decay

$\Rightarrow$ Test sharpens if the two are combined:

$$|V_u|^2 = 1.0000 \pm 0.0007 \qquad f_+(0) + V_{ud}$$
$$|V_u|^2 = 0.9999 \pm 0.0007 \qquad F_K/F_\pi + V_{ud}$$

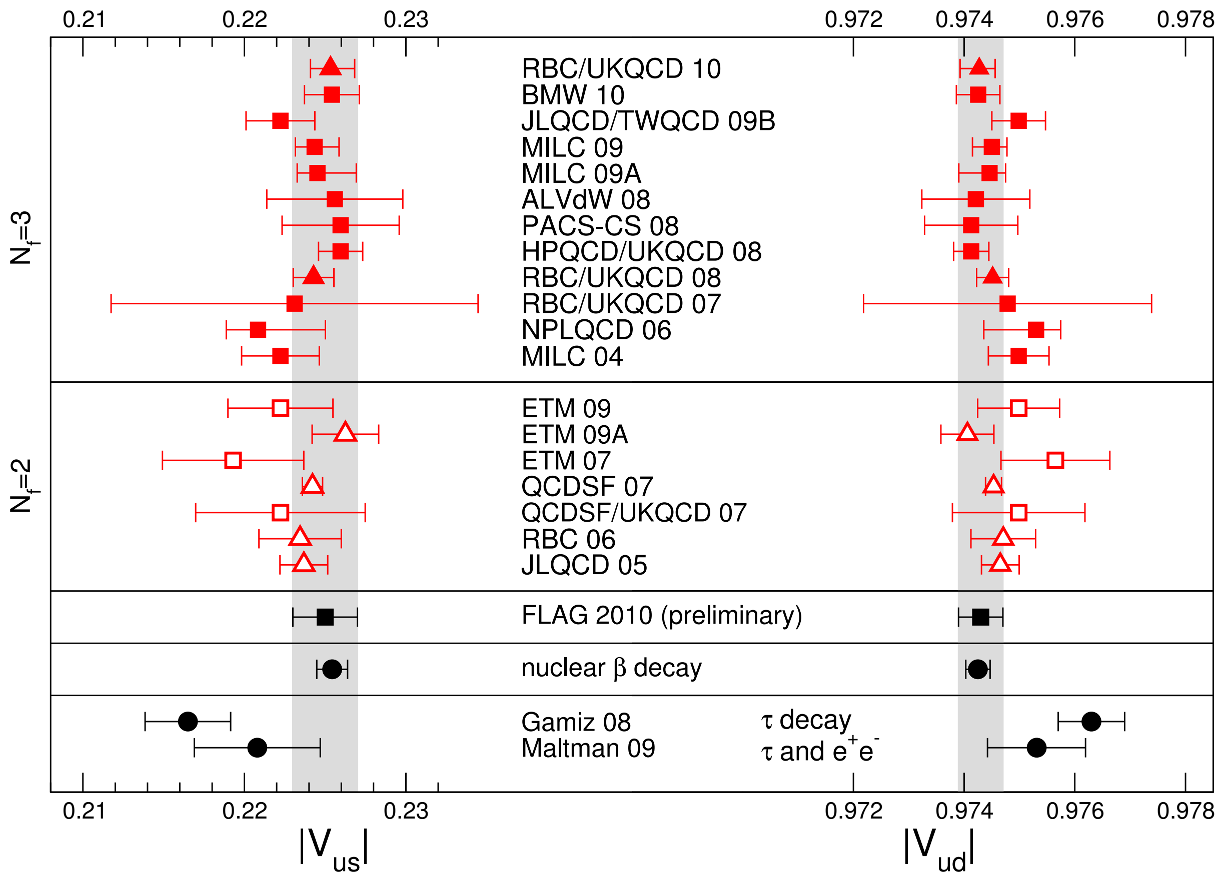
$\Uparrow$ Lattice $\quad$ $\Uparrow$ β-decay

$\Rightarrow$ Can impose $|V_u|^2 = 1$ as a constraint (SM)

	$\lvert V_{us}\rvert$	$\lvert V_{ud}\rvert$	$f_+(0)$	f_K/f_π
Lattice	0.225(2)	0.9743(4)	0.960(8)	1.193(11)
β decay	0.225(1)	0.9743(2)	0.960(5)	1.192(6)

FLAG review 2010 (preliminary)

Data on $|V_{us}|$ and $|V_{ud}|$ analyzed within the SM:



- Direct determination of $|V_{us}|$ from τ decay:

  Sort out the final states in the inclusive decay $\tau \to \nu$ + hadrons:

  $\Gamma = \Gamma(\tau \to \nu + \text{strange hadrons}) + \text{rest}$

  First term dominated by $|V_{us}|^2$, rest by $|V_{ud}|^2$

  Gamiz, Jamin, Pich, Prades, Schwab
  Maltman, Wolfe, Banerjee, Nugent, Roney

23. Puzzling results on $K_L \to \pi\mu\nu$

- Hadronic matrix element of weak current:

$$\langle K^0|\bar{u}\gamma^\mu s|\pi^-\rangle = (p_K+p_\pi)^\mu f_+(t)+(p_K-p_\pi)^\mu f_-(t)$$

- Scalar form factor $\sim \langle K^0|\partial_\mu(\bar{u}\gamma^\mu s)|\pi^-\rangle$

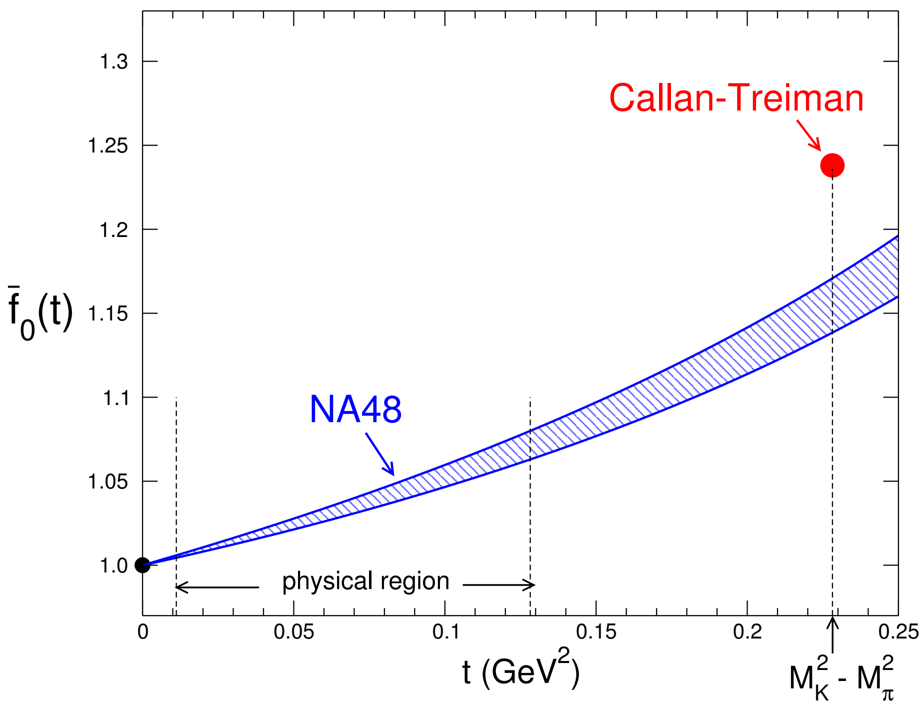
$$f_0(t) = f_+(t) + \frac{t}{M_K^2 - M_\pi^2} f_-(t)$$

- Low energy theorem — Callan & Treiman 1966

$$f_0(M_K^2 - M_\pi^2) = \frac{F_K}{F_\pi}\left\{1 + O(m_u, m_d)\right\} \simeq 1.19$$

$f_0(0) = f_+(0) \simeq 0.96$ relevant for determination of V_{us}

- Comparison with experiment



NA48, Phys. Lett. B647 (2007) 341 (141 authors, 2.3×10^6 events)

- Plot shows normalized scalar form factor

$$\bar{f}_0(t) = \frac{f_0(t)}{f_0(0)}$$

- CT relation in this normalization:

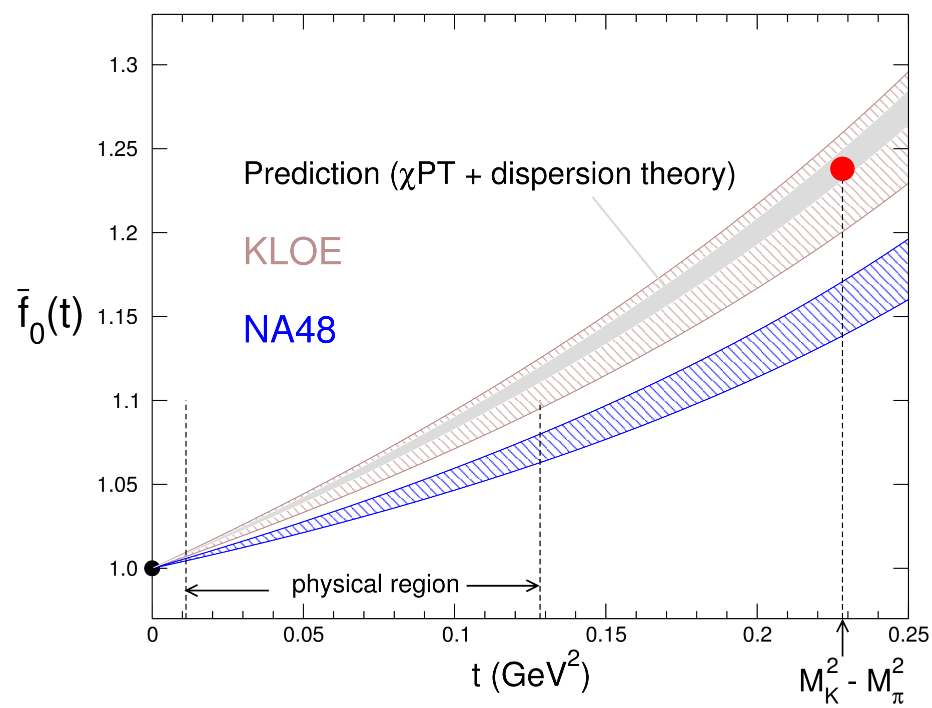
$$\bar{f}_0(M_K^2 - M_\pi^2) = \frac{F_K}{F_\pi f_+(0)} = 1.2446 \pm 0.0041$$

Bernard and Passemar 2008

- Implications

- NA48 data on $K_L \to \pi\mu\nu$ disagree with SM
- If confirmed, the implications are dramatic:
  $\Rightarrow W$ couples also to right-handed currents
  Bernard, Oertel, Passemar, Stern 2006
- There are not many places where the SM disagrees with observation, need to investigate these carefully
- At low energies, high precision is required

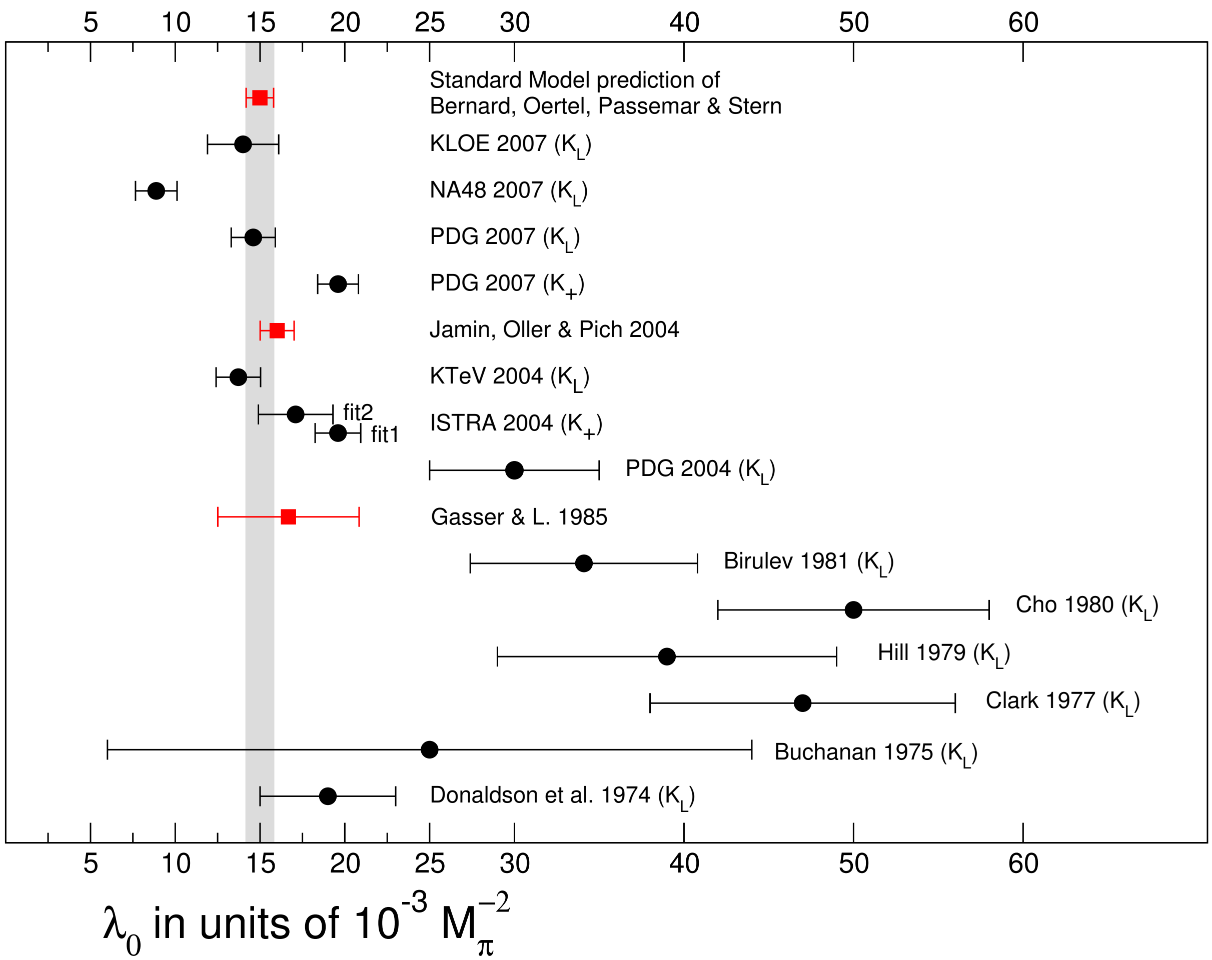
- New data from KLOE



I thank Emilie Passemar for some of the material shown in this figure

- History of the issue: data on the slope of the scalar form factor

$$f_0(t) = f_0(0)\left\{1 + \lambda_0\, t + \lambda_0'\, t^2 + O(t^3)\right\}$$



24. Concluding remarks

- These lectures focused on the low energy properties of the sector with zero baryon number: $N_B = \frac{1}{3}(N_u + N_d + N_s + N_c + N_b + N_t) = 0$. Moreover, only states with $N_c = N_b = N_t = 0$ were discussed.
- There is considerable progress in extending χPT to the sector with $N_B = 1$, as well as to nuclei, where $N_B = 2, 3 \ldots$
- Effective theory for heavy quark bound states
- Mesons with a heavy and a light quark
- Extension from QCD to QCD + QED

- Combine χPT with dispersion theory

  Example: form factors relevant for $K \to \pi \ell \nu$

  $f_0(t) = f_0(0)\left\{1 + \lambda_0\, t + \lambda_0'\, t^2 + \ldots\right\}$

  χPT: $\lambda_0 \leftrightarrow$ NLO, $\lambda_0' \leftrightarrow$ NNLO

  Dispersion theory implies very strong correlation between λ_0 and λ_0'

  Abbas, Ananthanarayan, Caprini, Imsong 2010

- Dispersive analysis of $\pi\pi$ and πK scattering, $\eta \to 3\pi$, . . .

If time permits, I can explain how dispersion theory can be used to extend the χPT result for the $\pi\pi$ scattering lengths to a model-independent prediction for mass and width of the σ meson