Non-perturbative Renormalization of Lattice QCD

Part V/V

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Continuum chiral WI's as normalisation conditions

- For chiral symmetry there is no conserved current with Wilson quarks.
- However: expect that chiral symmetry can be restored in the continuum limit!

⇒ [Bochicchio et al '85]: use continuum chiral Ward identities and impose them as normalisation condition at finite lattice spacing a!

Continuum chiral WI's as normalisation conditions

- Define chiral variations:

$$\delta^a_{\mathrm{A}}(\theta)\psi(x) = i\gamma_5 \tfrac{1}{2}\tau^a\theta(x)\psi(x), \qquad \delta^a_{\mathrm{A}}(\theta)\overline{\psi}(x) = \overline{\psi}(x) i\gamma_5 \tfrac{1}{2}\tau^a\theta(x)$$

- Derive formal continuum Ward identities assuming that the functional integral can be treated like an ordinary integral:

$$\Rightarrow \qquad \langle \delta^a_{\mathrm{A}}(\theta) O\rangle = \langle O \delta^a_{\mathrm{A}}(\theta) S\rangle,$$

$$\begin{aligned} \delta^a_{\mathrm{A}}(\theta) S &= -i\int \mathrm{d}^4 x\, \theta(x)\left(\partial_\mu A^a_\mu(x) - 2mP^a(x)\right) \\ A^a_\mu(x) &= \overline{\psi}(x)\gamma_\mu\gamma_5 \tfrac{1}{2}\tau^a\psi(x), \qquad P^a(x) = \overline{\psi}(x)\gamma_5\tfrac{1}{2}\tau^a\psi(x) \end{aligned}$$

Simplest chiral WI: the PCAC relation

- Shrink the region R to a point x:

$$\begin{aligned} \langle O_{\rm ext}\delta_{\rm A}^{a}(\theta)S\rangle &= 0 \\ \Rightarrow \quad \left\langle \partial_\mu A_\mu^a(x) O_{\rm ext}\right\rangle &= 2m \left\langle P^a(x) O_{\rm ext}\right\rangle \end{aligned}$$

- In the continuum the PCAC quark mass

$$m = \frac{\left\langle \partial_\mu A_\mu^a(x) O_{\rm ext}\right\rangle}{2\left\langle P^a(x) O_{\rm ext}\right\rangle}$$

must be independent of the choice for $O_{\rm ext}$, x, background field,...!

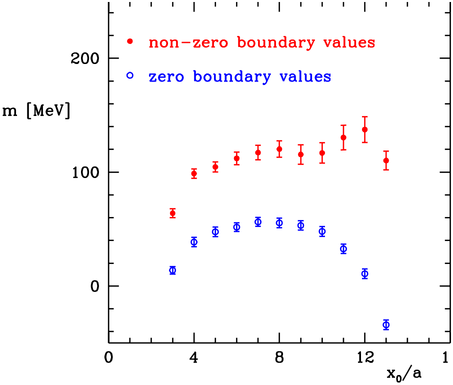
Need for O(a) improvement of Wilson quarks

O(a) artefacts can be quite large with Wilson quarks:

PCAC quark mass from SF correlation functions:

$$m = \frac{\partial_0 f_{\rm A}(x_0)}{2 f_{\rm P}(x_0)}$$

$8^3 \times 16$ lattice, quenched QCD, $a = 0.1\,\mathrm{fm}$, 2 different gauge background fields.



On-shell O(a) improvement

Recall Symanzik's effective continuum theory from lecture 1

$$S_{\rm eff} = S_0 + aS_1 + a^2 S_2 + \ldots, \qquad S_0 = S_{\rm QCD}^{\rm cont}$$

$$S_k = \int \mathrm{d}^4 x\, \mathcal{L}_{\rm k}(x)$$

where $\mathcal{L}_1$ is a linear combination of the fields:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad \overline{\psi}D_\mu D_\mu\psi, \qquad m\,\overline{\psi}\not{D}\psi, \quad m^2\overline{\psi}\psi, \qquad m\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\}$$

The action S_1 appears as insertion in correlation functions

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= \langle\phi_0(x_1)\ldots\phi_0(x_n)\rangle_{\rm con} \\ &\quad + a\int \mathrm{d}^4 y\ \langle\phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_1(y)\rangle_{\rm con} \\ &\quad + a\sum_{k=1}^{n}\langle\phi_0(x_1)\ldots\phi_1(x_k)\ldots\phi_0(x_n)\rangle_{\rm con} + \mathrm{O}(a^2) \end{aligned}$$

On-shell O(a) improvement (1)

Basic idea:

- Introduce counterterms to the *lattice* action and composite operators such that S_1 and ϕ_1 are cancelled in the effective theory
- As all physics can be obtained from on-shell quantitities (spectral quantitities like particle energies or correlation function where arguments are kept at non-vanishing distance) one may use the equations of motion to reduce the number of counterterms
- The contact terms which arise from having $y \approx x_i$ can be analysed in the OPE and are found to be of the same structure as the counterterms anyway contained in ϕ_1; this amounts to a redefinition of the counterterms in ϕ_1.
- After using the equations of motion one remains with:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad m^2\overline{\psi}\psi, \qquad m\,\mathrm{tr}\left\{F_{\mu\nu}F_{\mu\nu}\right\}$$

On-shell O(a) improvement (2)

1. On-shell O(a) improved Lattice action
   - The last two terms are equivalent to a rescaling of the bare mass and coupling ($m_{\rm q} = m_0 - m_{\rm cr}$):

$$\tilde{g}_0^2 = g_0^2(1 + b_g(g_0)am_{\rm q}), \qquad \tilde{m}_{\rm q} = m_{\rm q}(1 + b_{\rm m}(g_0)am_{\rm q})$$

   - The first term is the Sheikholeslami-Wohlert or clover term

$$S_{Wilson} \to S_{\rm Wilson} + iac_{\rm sw}(g_0)a^4 \sum_x \overline{\psi}(x)\sigma_{\mu\nu}\hat{F}_{\mu\nu}(x)\psi(x)$$

2. On-shell O(a) improved axial current and density:

$$\begin{aligned}
(A_{\rm R})^a_\mu &= Z_{\rm A}(\tilde{g}_0{}^2)(1 + b_{\rm A}(g_0)am_{\rm q})\left\{A^a_\mu + c_{\rm A}(g_0)\tilde{\partial}_\mu P^a\right\} \\
(P_{\rm R})^a &= Z_{\rm P}(\tilde{g}_0{}^2, a\mu)(1 + b_{\rm P}(g_0)am_{\rm q})P^a
\end{aligned}$$

On-shell O(a) improvement (3)

- There are 2 counterterms in the massless theory $c_{\rm sw}, c_{\rm A}$, the remaining ones ($b_g, b_{\rm m}, b_{\rm A}, b_{\rm P}$) come with $am_{\rm q}$.
- Note: all counterterms are absent in chirally symmetric regularisations!

$\Rightarrow$ turn this around: impose chiral symmetry to determine $c_{\rm sw}, c_{\rm A}$ non-perturbatively:

- define bare PCAC quark masses from SF correlation functions

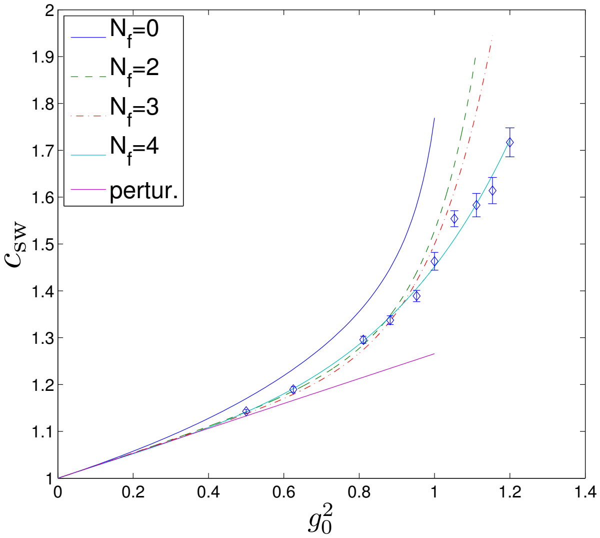
$$m_{\rm R} = \frac{Z_{\rm A}(1 + b_{\rm A}\, am_{\rm q})}{Z_{\rm P}(1 + b_{\rm P}\, am_{\rm q})} m, \qquad m = \frac{\tilde{\partial}_0 f_{\rm A}(x_0) + c_{\rm A}\, a\partial_0^*\partial_0 f_{\rm P}(x_0)}{f_{\rm P}(x_0)}$$

- At fixed g_0 and $am_{\rm q} \approx 0$ define 3 bare PCAC masses $m_{1,2,3}$ (e.g. by varying the gauge boundary conditions) and impose

$$m_1(c_{\rm sw}, c_{\rm A}) = m_2(c_{\rm sw}, c_{\rm A}), \quad m_1(c_{\rm sw}, c_{\rm A}) = m_3(c_{\rm sw}, c_{\rm A}) \Rightarrow c_{\rm sw}, c_{\rm A}$$

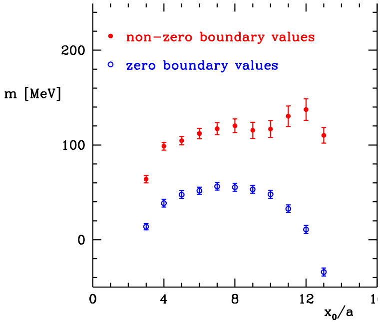
SF b.c.'s $\Rightarrow$ high sensitivity to $c_{\rm sw}$ & simulations near chiral limit

Results for $c_{\rm sw}$, $N_{\rm f} = 4$ [ALPHA '09]



On-shell O(a) improvement (4)

Before and after O(a) improvement (PCAC masses from SF correlation functions, $8^3 \times 16$ lattice)



Quenched result for the charm quark mass [ALPHA '02]

- The RGI charm quark mass can be defined in various ways
  - starting from the subtracted bare quark mass $m_{\rm q,c} = m_{0,c} - m_{\rm cr}$
  - starting from the average strange-charm PCAC mass m_{sc}
  - starting from the PCAC mass m_{cc} for a hypothetical mass degenerate doublet of quarks.
- Tune bare charm quark mass to match the D_s meson mass
- Obtain the corresponding O(a) improved RGI masses:

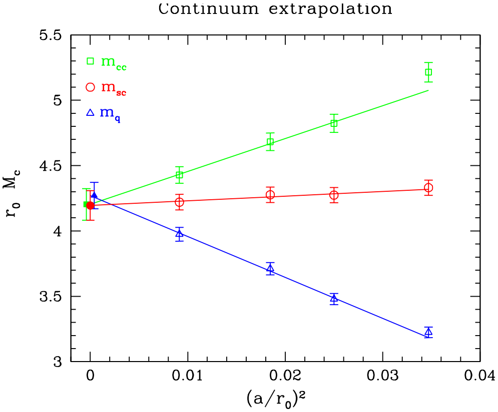
$$
\begin{aligned}
r_0 M_c|_{m_{sc}} &= Z_M r_0 \Big\{ 2m_{sc} \left[1 + (b_{\rm A} - b_{\rm P})\tfrac{1}{2}(am_{\rm q,c} + am_{\rm q,s})\right] \\
&\qquad - m_{\rm s} \left[1 + (b_{\rm A} - b_{\rm P}) am_{\rm q,s})\right] \Big\}, \\
r_0 M_c|_{m_{\rm c}} &= Z_M r_0 m_{\rm c} \left[1 + (b_{\rm A} - b_{\rm P}) am_{\rm q,c}\right], \\
r_0 M_c|_{m_{\rm q,c}} &= Z_M Z r_0 m_{\rm q,c} \left[1 + b_{\rm m} am_{\rm q,c}\right].
\end{aligned}
$$

- N.B.: all O(a) counterterms are known non-perturbatively in the quenched case!

Continuum extrapolation of the quenched RGI charm quark mass

Continuum extrapolation:

$$
\begin{aligned}
r_0 M_c &= A + B(a^2/r_0^2) \\
r_0 &= 0.5\,\mathrm{fm} \\
M_c &= 1.654(45)\,\mathrm{GeV} \\
\overline{m}_c^{\overline{\mathrm{MS}}}(\overline{m}_c) &= 1.301(34)\,\mathrm{GeV}
\end{aligned}
$$



Summary On-shell O(a) improvement

After O(a) improvement:

- The ambiguity in m_{cr} is reduced to O(a^2)
- Axial current normalisation can be defined up to O(a^2)
- Results exist for $c_{\mathrm{sw}}, c_{\mathrm{A}}$ for quenched and $N_{\mathrm{f}} = 2, 3, 4$ and various gauge actions
- On-shell O(a) improvement seems to work; rather economical for spectral quantities (e.g. hadron masses): just need c_{sw}!
- Improvement of quark bilinear operators feasible, four-quark operators difficult
- Non-degenerate quark masses: rather complicated, proliferation of b-coefficients [Bhattacharya et al '99 ff];
- However: for small quark masses and fine lattices am_{q} is small (a few percent at most) and perturbative estimates of improvement coefficients may be good enough!

The Schrödinger functional and O(a) improvement

The presence of the boundaries induces additional O(a) effects:

- counterterms must be local fields of dimension 4 integrated over the boundaries $x_0 = 0, T$:
- pure gauge theory:

$$\int \mathrm{d}^3\mathbf{x}\ \mathrm{tr}\{F_{0k}(x)F_{0k}(x)\}, \quad \int \mathrm{d}^3\mathbf{x}\ \mathrm{tr}\{F_{kl}(x)F_{kl}(x)\} = 0\ (\rightarrow \text{b.c.'s})$$

- with fermions:

$$\int \mathrm{d}^3\mathbf{x}\,\overline{\psi}(x)\gamma_0 D_0\psi(x), \quad \int \mathrm{d}^3\mathbf{x}\,\overline{\psi}(x)\gamma_k D_k\psi(x),$$

  eliminate 2nd counterterm by equation of motion

$\Rightarrow$ all boundary O(a) effects can be cancelled by 2 counterterms with coefficients $c_{\mathrm{t}}, \tilde{c}_{\mathrm{t}}$!

- In practice use perturbation theory and vary the coefficients in simulations to assess their impact on observables.

Automatic O(a) improvement of massless Wilson quarks [Frezzotti, Rossi '03]

- Assume $m_{\rm PCAC} = 0$, finite volume without boundaries:

$\Rightarrow$ Symanziks effective continuum action (using eqs. of motion):

$$S_{\rm eff} = S_0 + aS_1 + \ldots, \quad S_0 = \int \mathrm{d}^4x\, \overline{\psi} \not{D} \psi, \; S_1 = c \int \mathrm{d}^4x\, \overline{\psi} \sigma_{\mu\nu} F_{\mu\nu} \psi$$

- cutoff dependence of lattice correlation functions:

$$\langle O \rangle = \langle O \rangle^{\rm cont} - a \langle S_1 O \rangle^{\rm cont} + a \langle \delta O \rangle^{\rm cont} + O(a^2).$$

δO are O(a) counterterms to the composite fields in O, e.g.

$$\begin{aligned} O &= V_\mu^a(x) A_\nu^b(y) \\ \delta O &= c_{\rm V}\, i\partial_\nu T_{\mu\nu}^a(x) A_\nu^a(y) + V_\mu^a(x) c_{\rm A} \partial_\nu P^b(y) \end{aligned}$$

Automatic O(a) improvement of massless Wilson quarks

- Introduce a γ_5-transformation (non-anomalous for even numbers of quarks):

$$\psi \rightarrow \gamma_5 \psi, \qquad \overline{\psi} \rightarrow -\overline{\psi}\gamma_5$$

- transform Symanzik's effective action and O(a) counterterms

$$S_0 \quad \rightarrow \quad S_0, \qquad S_1 \rightarrow -S_1$$

- Composite operators can be decomposed in γ_5-even and -odd parts:

$$\begin{aligned} O &= O_+ + O_- \\ O_\pm &\rightarrow \pm O \quad \Rightarrow \quad \delta O_\pm \rightarrow \mp \delta O_\pm \end{aligned}$$

- Hence for γ_5-even O_+ one finds

$$\begin{aligned} \langle O_+\rangle^{\mathrm{cont}} &= \langle O_+\rangle^{\mathrm{cont}} \\ \langle O_+ S_1\rangle^{\mathrm{cont}} &= -\langle O_+ S_1\rangle^{\mathrm{cont}} = 0 \\ \langle \delta O_+\rangle^{\mathrm{cont}} &= -\langle \delta O_+\rangle^{\mathrm{cont}} = 0 \\ \Rightarrow \quad \langle O_+\rangle &= \langle O_+\rangle^{\mathrm{cont}} + O(a^2) \end{aligned}$$

- while for γ_5-odd O_- one gets

$$\begin{aligned} \langle O_-\rangle^{\mathrm{cont}} &= -\langle O_-\rangle^{\mathrm{cont}} = 0 \\ \langle O_- S_1\rangle^{\mathrm{cont}} &= \langle O_- S_1\rangle^{\mathrm{cont}} \\ \langle \delta O_-\rangle^{\mathrm{cont}} &= \langle \delta O_-\rangle^{\mathrm{cont}} \\ \Rightarrow \quad \langle O_-\rangle &= -a\langle O_- S_1\rangle^{\mathrm{cont}} + a\langle \delta O_-\rangle^{\mathrm{cont}} + O(a^2) \end{aligned}$$

$\Rightarrow$ γ_5-even observables are automatically $\mathrm{O}(a)$ improved, while γ_5-odd observables vanish up to $\mathrm{O}(a)$ terms.

<u>Remarks:</u>

- The cutoff effects are located in the γ_5-odd components. These can be easily identified and projected out for any lattice field, and the elimination of cutoff effects is then "automatic".
- In fermion regularisation with an exact chiral symmetry (Ginsparg-Wilson quarks) the γ_5-odd fields vanish identically $\Rightarrow$ no need to project out the odd components.
- A_μ^a and P^a have opposite γ_5-parity!

$$\Rightarrow \qquad \langle \partial_\mu A_\mu^a(x) O_{\mathrm{even}} \rangle = 2\, \underbrace{m_{\mathrm{PCAC}}}_{\mathrm{O}(a)} \underbrace{\langle P^a(x) O_{\mathrm{even}} \rangle}_{\mathrm{O}(a)} = O(a^2)$$

i.e. the critical mass need only be defined up to an $\mathrm{O}(a)$ ambiguity.

SF schemes with Wilson quarks and O(a) improvement

- previous discussion:
  γ_5-even observables computed with Wilson quarks in a finite volume (with some type of periodic boundary conditions) are automatically O(a) improved at zero quark mass
- SF coupling and renormalization factors are computed at zero quark mass.
  distinguish 3 sources for O(a) effects:
  1. O(a) boundary effects (expected in any case!); can be cancelled by inclusion of boundary O(a) counterterms
  2. from the bulk action; are cancelled by including the SW/clover term
  3. from the composite operators; can be cancelled by including O(a) counterterms determined from chiral Ward identities; difficult for 4-quark operators!

Question: Why do the bulk O(a) counterterms not vanish in the chiral limit?

- Problem: the γ_5 field transformation switches the projectors of the quark b.c.'s:

$$P_\pm\gamma_5 = \gamma_5 P_\mp$$

The boundary conditions, like mass terms, break chiral symmetry and define a direction in chiral flavour space.

$\Rightarrow$ the γ_5 transformation yields inequivalent correlation functions even in the chiral limit,

$$\langle O\rangle_{(P_\pm)} \to \langle O'\rangle_{(P_\mp)}$$

- Possible solution: Give a flavour structure to the γ_5-transformation, for $N_\mathrm{f} = 2$:

$$\psi \to \gamma_5\tau^1\psi, \qquad \overline{\psi} \to -\overline{\psi}\gamma_5\tau^1,$$

and change quark boundary projectors, such that they commute with $\gamma_5\tau^1$, e.g.

$$\mathcal{P}_\pm = \tfrac{1}{2}(1 \pm \gamma_0\tau^3), \qquad Q_\pm = \tfrac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3),$$

SF boundary conditions and chiral rotations

Consider isospin doublets ψ' and $\overline{\psi}'$ satisfying homogeneous SF boundary conditions ($P_\pm = \frac{1}{2}(1 \pm \gamma_0)$),

$$P_+\psi'(x)|_{x_0=0} = 0, \qquad P_-\psi'(x)|_{x_0=T} = 0,$$
$$\overline{\psi}'(x)P_-|_{x_0=0} = 0, \qquad \overline{\psi}'(x)P_+|_{x_0=T} = 0.$$

perform a chiral field rotation,

$$\psi' = \exp(i\alpha\gamma_5\tau^3/2)\psi, \qquad \overline{\psi}' = \overline{\psi}\exp(i\alpha\gamma_5\tau^3/2),$$

the rotated fields satisfy chirally rotated boundary conditions

$$P_+(\alpha)\psi(x)|_{x_0=0} = 0, \qquad P_-(\alpha)\psi(x)|_{x_0=T} = 0,$$
$$\overline{\psi}(x)\gamma_0 P_-(\alpha)|_{x_0=0} = 0, \qquad \overline{\psi}(x)\gamma_0 P_+(\alpha)|_{x_0=T} = 0,$$

with the projectors

$$P_\pm(\alpha) = \frac{1}{2}\left[1 \pm \gamma_0 \exp(i\alpha\gamma_5\tau^3)\right].$$

SF boundary conditions and chiral rotations

$$P_\pm(\alpha) = \tfrac{1}{2}\left[1 \pm \gamma_0 \exp(i\alpha\gamma_5\tau^3)\right].$$

Special cases of $\alpha = 0, \pi/2$:

$$P_\pm(0) = P_\pm, \qquad P_\pm(\pi/2) \equiv \tilde{Q}_\pm = \tfrac{1}{2}(1 \pm i\gamma_0\gamma_5\tau^3),$$

The chiral rotation thus introduces a mapping between correlation functions:

$$\langle O[\psi,\bar\psi]\rangle_{(P_\pm)} = \langle \tilde{O}[\psi,\bar\psi]\rangle_{(P_\pm(\alpha))}$$

$$\text{with} \quad \tilde{O}[\psi,\bar\psi] \;=\; O\left[\exp(i\alpha\gamma_5\tau^3/2)\psi, \bar\psi \exp(i\alpha\gamma_5\tau^3/2\right]$$

where boundary quark and anti-quark fields are included by replacing

$$\bar\zeta(\mathbf{x}) \leftrightarrow \bar\psi(0,\mathbf{x})P_+ \qquad \zeta(\mathbf{x}) \leftrightarrow P_-\psi(0,\mathbf{x})$$

The chirally rotated Schrödinger functional

For even N_f there is an implementation of the χSF with Wilson quarks [S.S. '05–10]

- implements automatic bulk O(a) improvement
- related to standard SF by chiral rotation
- Alternative to chiral Ward identities:
  - Universality relations $\Rightarrow$ determinations of $Z_{\rm A}$ etc.
  - Improvement conditions: demand $\gamma_5\tau^1$ odd quantities to vanish!

Universality checks

- The dictionary tells us that e.g.

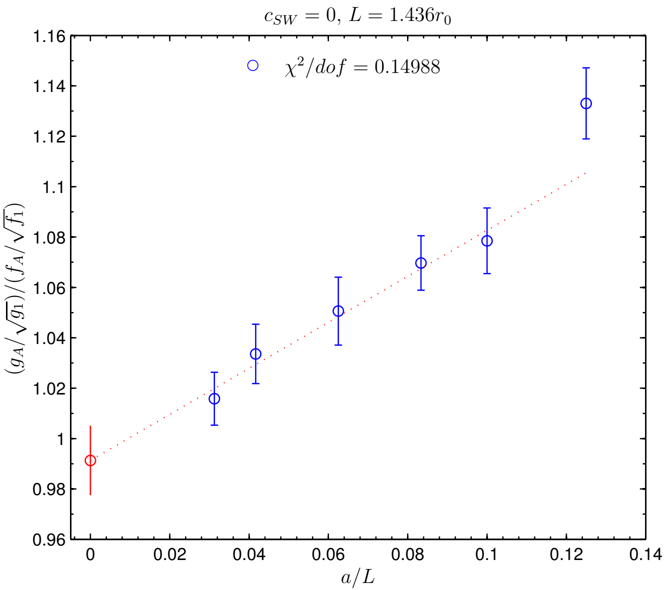
$$f_{\mathrm{A}}(x_0) = g_A^{uu'}(x_0), \qquad f_{\mathrm{P}}(x_0) = g_{\mathrm{P}}^{ud}(x_0)$$

$\Rightarrow$ should hold for renormalized correlation functions! Here: check that

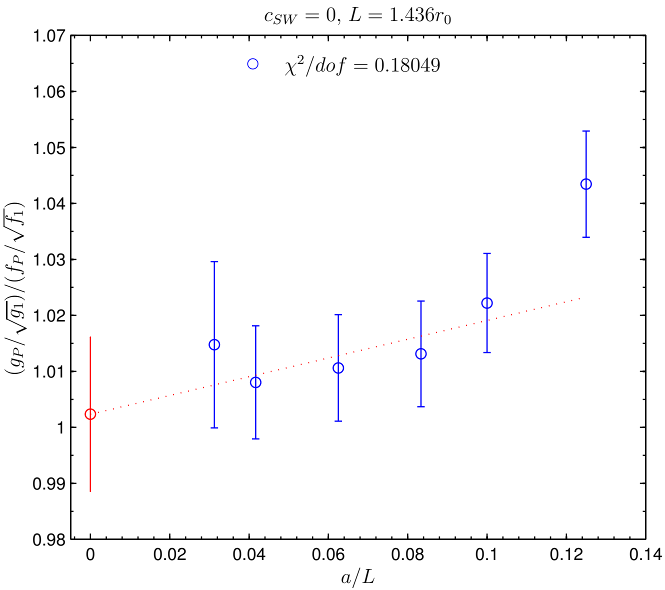
$$\frac{\tilde{Z}_{\mathrm{A}} f_{\mathrm{A}}(T/2)}{\sqrt{f_1}} = \frac{\tilde{Z}_{\mathrm{A}} g_{\mathrm{A}}^{uu'}(T/2)}{\sqrt{g_1^{uu'}}}, \qquad \frac{Z_{\mathrm{P}} f_{\mathrm{P}}(T/2)}{\sqrt{f_1}} = \frac{Z_{\mathrm{P}} g_{\mathrm{P}}^{uu'}(T/2)}{\sqrt{g_1^{uu'}}},$$

- note: $\tilde{Z}_{\mathrm{A}}$ is not the Z_{A} to normalize the axial current canonically.

Universality checks, axial current [S. & Leder '10]



Universality checks, axial density [S. & Leder '10]



Checks of automatic O(a) improvement [S. & Leder '10]

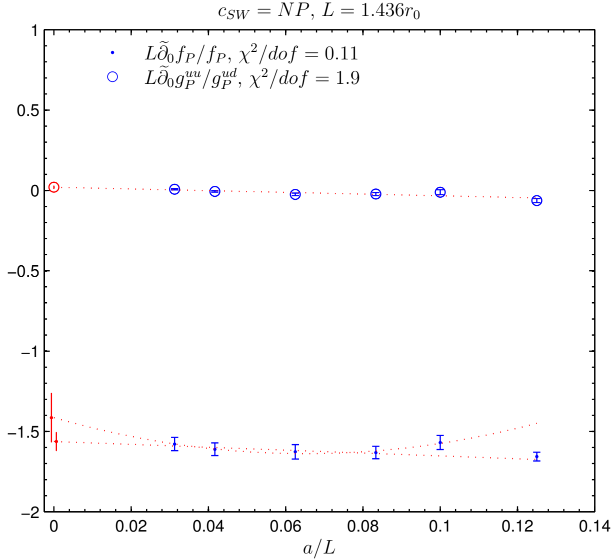
Check that bulk O(a) counterterms to even operators become irrelevant (up to O(a^2) effects)

- choose $c_{\rm A}$-counterterm in improved axial current $A_\mu + c_{\rm A} a \partial_\mu P$. Should find that

$$\frac{L\partial_0 f_{\rm P}(x_0)}{f_{\rm P}(x_0)} = \mathrm{O}(1), \qquad \frac{L\partial_0 g_{\rm P}^{uu'}(x_0)}{g_{\rm P}^{ud}(x_0)} = \mathrm{O}(a)$$

Note: renormalization constants drop out in ratios!

Checks of automatic O(a) improvement, counterterm $\propto c_{\mathrm{A}}$ [S. & Leder '10]



Determination of $Z_{\mathrm{A,V}}$ [S. & Leder '10]

Use universality to determine finite Z-factors otherwise determined from axial Ward identities:

- Chiral rotation of currents depends on the flavour structure:

$$f_{\mathrm{A}} = g_{\mathrm{A}}^{uu'} = -i g_{\mathrm{V}}^{ud}$$

$\Rightarrow$ axial current transforms either to a vector or to an axial current

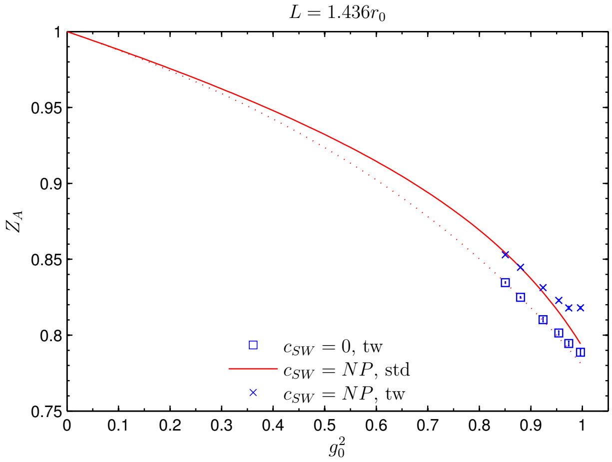
- universality: correctly renormalized correlation functions should be equal in continuum limit
- assume universality to obtain correct renormalization constant

$$1 = \frac{Z_{\mathrm{A}} g_{\mathrm{A}}^{uu'}(T/2}{-i Z_{\mathrm{V}} g_{\mathrm{V}}^{ud}(T/2)} \quad \Rightarrow \quad Z_{\mathrm{A}}/Z_{\mathrm{V}}$$

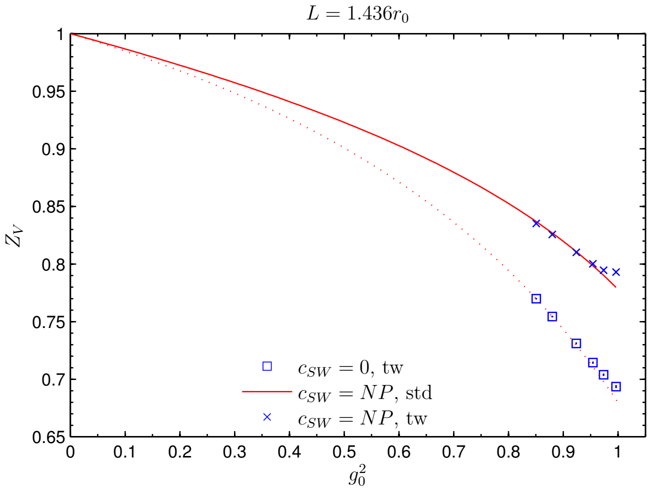
- vector current: compare conserved to local current:

$$1 = \frac{Z_{\mathrm{V}} g_{\mathrm{V}}^{ud}(T/2}{g_{\tilde{V}}^{ud}(T/2)} \quad \Rightarrow \quad Z_{\mathrm{V}}$$

Determination of $Z_{\rm A}$ [S. & Leder '10]



Determination of $Z_{\rm V}$ [S. & Leder '10]



The gradient flow and finite volume couplings (1)

[Lüscher '10 ff] Starting from a given gauge potential $A_\mu(x)$ evolve in "flow time" t:

$$\begin{aligned} \frac{d}{dt} B_\mu(x,t) &= D_\nu G_{\nu\mu}(x,t), \qquad B_\mu(x,0) = A_\mu(x), \\ G_{\mu\nu} &= [D_\mu, D_\nu], \qquad D_\mu = \partial_\mu + [B_\mu, \cdot] \end{aligned}$$

- The force term on the RHS is the gradient of the Yang-Mills action;

⇒ The flow drives the gauge field towards the classical minimum configuration of the Yang-Mills action, hence towards smoother fields.

The gradient flow and finite volume couplings (2)

[Lüscher & Weisz '10 ff]

- The flow time t can be interpreted as coordinate of a 5th dimension, i.e. theory lives in 5-dimensional half space with Dirichlet boundary condition at $t = 0$
- Renormalization: correlation functions in the five dimensional theory are finite once the 4-d theory at $t = 0$ has been renormalized.

$\Rightarrow$ no new divergences are generated by the flow!

Example: the expectation value of the energy density

$$\langle E(t) \rangle = \frac{1}{2} \langle \operatorname{tr} [G_{\mu\nu}(x) G_{\mu\nu}(x)] \rangle$$

is well defined and finite for $t > 0$ provided g_0 is renormalized as usual.

Perturbation theory in $\overline{\mathrm{MS}}$-coupling $\alpha(\mu = 1/\sqrt{8t})$ [Lüscher '10]

$$\langle E(t) \rangle = \frac{3}{4\pi t^2} \alpha(\mu) \{1 + k_1 \alpha(\mu) + ...\}, \quad k_1 = 1.0978 + 0.0075 \times N_\mathrm{f}$$

Non-perturbatively:

- implement flow on the lattice for link variables:

$$\frac{d}{dt} V_\mu(x,t) = -g_0^2 \{\partial_{x,\mu} S_w\} V_\mu(x,t), \qquad V_\mu(x,0) = U_\mu(x)$$

- S_w = Wilson's plaquette action ⇒ "Wilson flow"

Marry Wilson flow with finite volume

Applications of the Wilson flow:

- Scale setting, t_0 [Lüscher '10]:

$$t^2 \langle E(t) \rangle |_{t=t_0} = 0.3$$

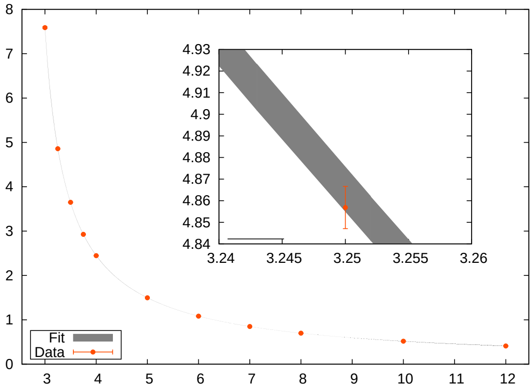
- Finite volume schemes: take $\mu = 1/\sqrt{8t}$ and $1/L$ in fixed proportion, $\mu = c/L$ with constant c;
  use
  - periodic boundary conditions [Fodor et al '12];
  - SF boundary conditions [Fritzsch & Ramos '12];
  - twisted periodic boundary conditions [Ramos '13];

$SU(2)$ YM running coupling with twisted b.c.'s [A. Ramos, Lattice 2013]

- Simulations for $L/a = 10, 12, 15, 18, 20, 24, 30, 36$ at $\beta \in [2.75, 12]$.
- Modest statistics: 2048 independent measurements of g^2_{TGF}.
- Between 0.15-0.25% precision in g^2_{TGF} *for all* L/a.
- Example: $L/a = 36$

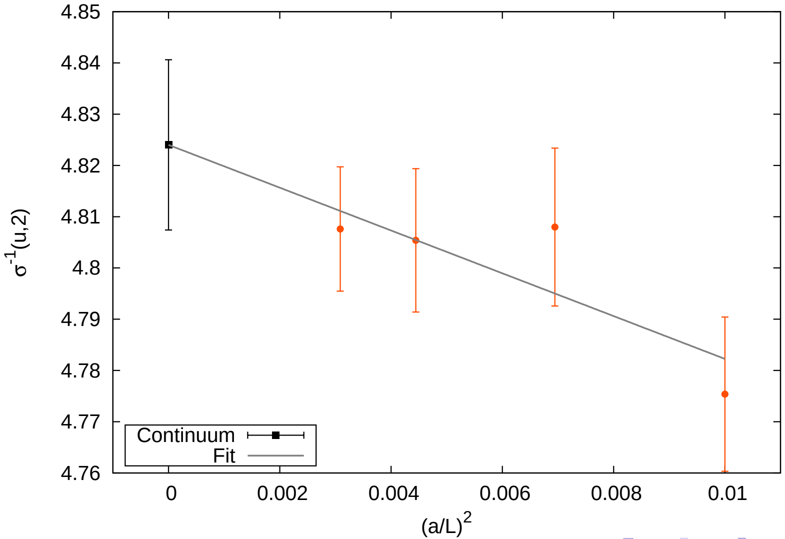
$SU(2)$ YM running coupling [A. Ramos, Lattice 2013]

β	$g^2_{TGF}(L)$
12.0	0.41078(64)
10.0	0.51809(83)
8.0	0.6987(11)
7.0	0.8497(13)
6.0	1.0819(18)
5.0	1.4968(25)
4.0	2.4465(44)
3.75	2.9277(54)
3.5	3.6494(69)
3.25	4.8568(99)
3.0	7.587(20)
2.9	10.610(32)
2.8	16.752(47)
2.75	22.168(59)



Step scaling function [A. Ramos, Lattice 2013]

- Modest cutoff effects. Starting recursion with $u = 7.5$.



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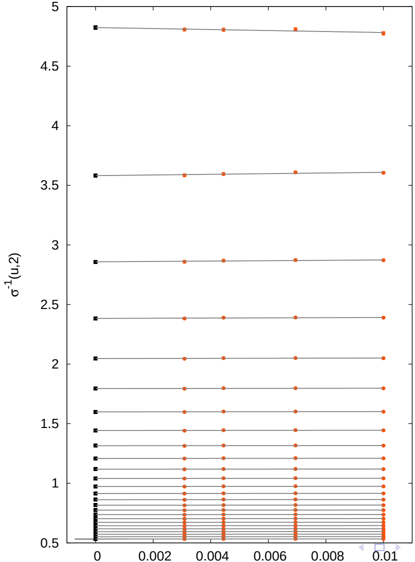
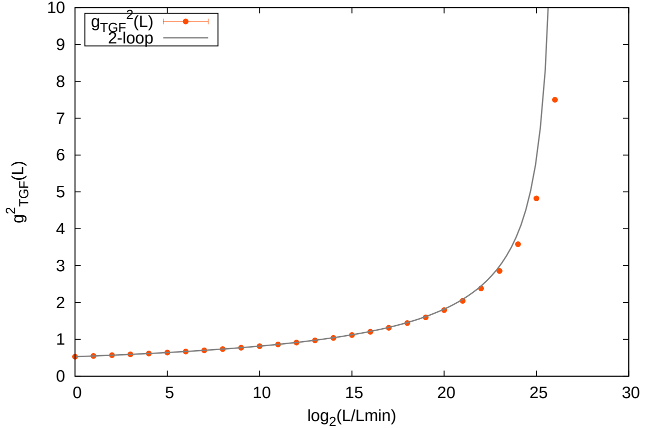


$g^2_{TGF}(L)$ for pure gauge $SU(2)$ [A. Ramos, Lattice 2013]



Since $\Lambda = \mu(b_0 g^2(\mu))^{-b1/2b_0^2} e^{-1/2b_0 g^2(\mu)} e^{-\int_0^{g^2(\mu)}\left\{\frac{1}{\beta(x)}+\frac{1}{b_0 x^3}-\frac{b_1}{b_0^2 x}\right\}}$
and $\mu = 1/cL$.

$$\Lambda L_{\mathrm{max}} = 1.509(44) \quad (@g^2_{TGF}(L) = 1.7948(93))$$

Conclusions & Outlook Wilson flow

The Wilson flow allows for definition of renormalized quantities which

- are very precisely measurable (small statistical errors);
- can be used to set the scale of lattice simulations (t_0, w_0, cf. Sommer Lattice '13);
- couple to slow modes in algorithms $\Rightarrow$ trace autocorrelations in HMC;
- can be combined with renormalization in finite volume & Schrödinger functional;

$\Rightarrow$ expect precise results for α_s over next 2 years!

Generalization to include fermion fields [Lüscher '13]

- may lead to better renormalization/improvement conditions;
- may solve some notorious problems with power divergent renormalizations e.g. 4-quark operators, energy-momentum tensor (cp. del Debbio et al. '13);
- ....

$\Rightarrow$ Stay tuned!