Non-perturbative Renormalization of Lattice QCD

Part IV/V

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- The coupling in the Schrödinger functional scheme
- SF schemes for composite operators ($\rightarrow$ mass renormalization)
- Step scaling functions and how to get them from lattice approximants
- Some results by the ALPHA collaboration
- Symmetries and Ward identities
- Wilson quarks and chiral Ward identities

Definition of the SF coupling [Lüscher et al. '92]

- Choose abelian and spatially constant boundary gauge fields:

$$C_k = \frac{i}{L}\begin{pmatrix}\phi_1 & 0 & 0\\ 0 & \phi_2 & 0\\ 0 & 0 & \phi_3\end{pmatrix}, \qquad C'_k = \frac{i}{L}\begin{pmatrix}\phi'_1 & 0 & 0\\ 0 & \phi'_2 & 0\\ 0 & 0 & \phi'_3\end{pmatrix}, \qquad k=1,2,3$$

- with angles taken to be linear functions of a parameter η:

$$\begin{aligned}\phi_1 &= \eta - \tfrac{\pi}{3}, & \phi'_1 &= -\phi_1 - \tfrac{4\pi}{3},\\ \phi_2 &= -\tfrac{1}{2}\eta, & \phi'_2 &= -\phi_3 + \tfrac{2\pi}{3},\\ \phi_3 &= -\tfrac{1}{2}\eta + \tfrac{\pi}{3}, & \phi'_3 &= -\phi_2 + \tfrac{2\pi}{3}.\end{aligned}$$

- The gauge action has an absolute minimum for:

$$B_0 = 0, \qquad B_k = \left[x_0 C'_k + (L - x_0) C_k\right]/L, \quad k = 1,2,3.$$

i.e. other gauge fields with the same action must be gauge equivalent to B_μ

Definition of the SF coupling

- Define the effective action of the induced background field

$$\Gamma[B] = -\ln \mathcal{Z}[C, C']$$

- In perturbation theory the effective action has the expansion

$$\Gamma[B] \quad \sim \quad g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + O(g_0^2)$$

- Definition of the SF coupling:

$$\bar{g}^2(L) = \left.\frac{\partial_\eta \Gamma_0[B]|_{\eta=0}}{\partial_\eta \Gamma[B]|_{\eta=0}}\right|_{m_{\mathrm{q,i}}=0} \qquad \Rightarrow \qquad \bar{g}^2(L) = g_0^2 + \mathrm{O}(g_0^4)$$

- b.c.'s induce a constant colour electric field:

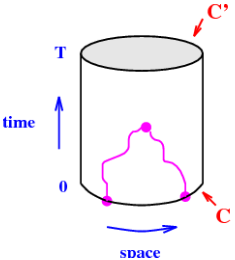
$$G_{0k} = \partial_0 B_k = \frac{C_k - C_k'}{L}$$

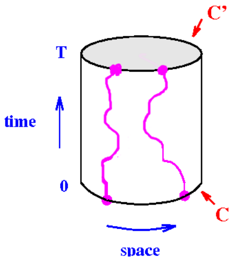
$\Rightarrow$ The coupling is defined as “response coefficient” to a variation of a constant colour electric field.

Example: renormalisation of $P^a = \overline{\psi}\gamma_5 \frac{\tau^a}{2}\psi$:

- In this case we set $C_k = C'_k = 0$, i.e. trivial background field $B = 0$
- Define correlation functions

$$f_{\rm P}(x_0) = -\tfrac{1}{3}\langle \mathcal{O}^a P^a(x)\rangle, \qquad f_1 = -\tfrac{1}{3L^6}\langle \mathcal{O}^a \mathcal{O}'^a\rangle$$





Renormalisation of operators in the SF scheme (2)

- Renormalised correlation functions:

$$f_{\mathrm{P,R}}(x_0) = Z_\zeta^2 Z_P f_{\mathrm{P}}(x_0), \qquad f_{1,R} = Z_\zeta^4 f_1,$$

set $T = L$, $m = 0$, $x_0 = L/2$, and impose

$$Z_{\mathrm{P}}(g_0, L/a)\frac{f_{\mathrm{P}}(L/2)}{\sqrt{f_1}} = \left.\frac{f_{\mathrm{P}}(L/2)}{\sqrt{f_1}}\right|_{g_0=0}$$

- similarity with MOM schemes: the renormalised amplitude at $\mu = L^{-1}$ equals its tree-level expression
- The ratio is formed to cancel any Z_ζ.
- definition of running quark mass: $\overline{m}(L) = Z_{\mathrm{P}}^{-1}(L)m$.

Step Scaling Functions

- The aim is to construct the Step Scaling Functions $\sigma(u)$ and $\sigma_{\rm P}(u)$:

$$\begin{aligned} \sigma(u) &= \bar{g}^2(2L)\big|_{u=\bar{g}^2(L)}, \\ \sigma_{\rm P}(u) &= \lim_{a\to 0} \frac{Z_{\rm P}(g_0, 2L/a)}{Z_{\rm P}(g_0, L/a)}\bigg|_{u=\bar{g}^2(L)} \end{aligned}$$

- These are related to the usual RG functions:

$$\int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\mathrm{d}g}{\beta(g)} = \ln 2 \qquad \sigma_{\rm P}(u) = \exp \int_{\sqrt{\sigma(u)}}^{\sqrt{u}} \frac{\tau(g)}{\beta(g)} \mathrm{d}g$$

- One thus considers a change of scale by a finite factor $s = 2$; RG functions tell us what happens for infinitesimal scale changes.

Lattice approximants $\Sigma(u, a/L)$ for $\sigma(u)$

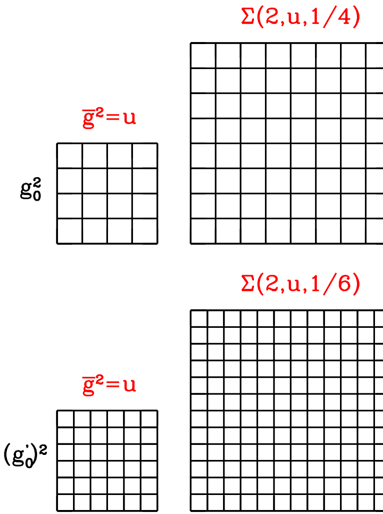
- choose g_0 and $L/a = 4$, measure $\bar{g}^2(L) = u$ (this sets the value of u)
- double the lattice and measure

$$\Sigma(u, 1/4) = \bar{g}^2(2L)$$

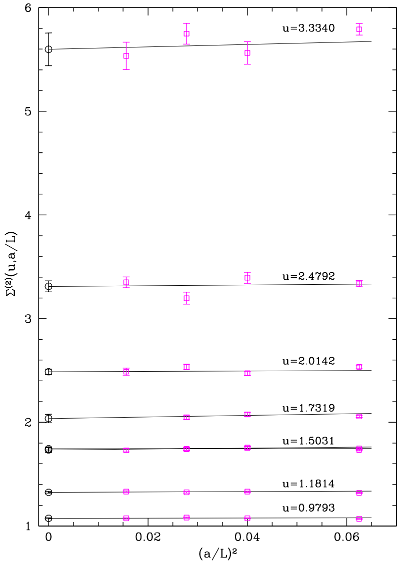
- now choose $L/a = 6$ and tune g_0' such that $\bar{g}^2(L) = u$ is satisfied
- double the lattice and measure

$$\Sigma(u, 1/6) = \bar{g}^2(2L)$$

- and so on ...

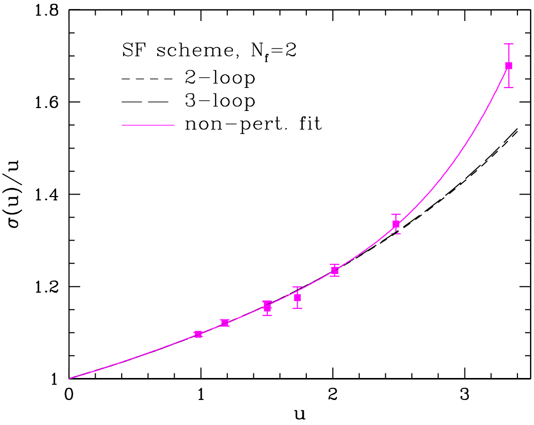


Continuum extrapolation of the SSF [ALPHA '05]



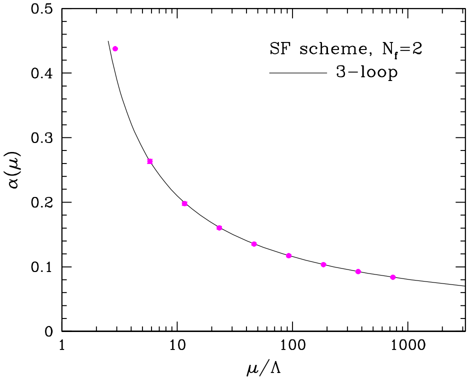
The SSF in the continuum limit

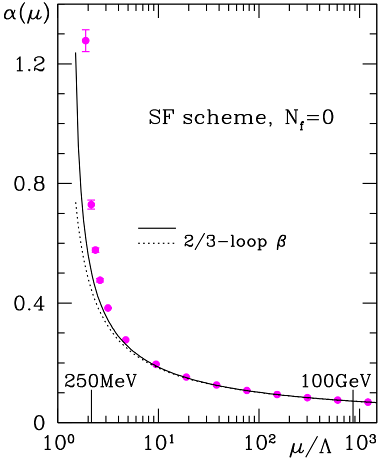
[ALPHA coll., Della Morte et al '05]



The running of the SF coupling

[ALPHA coll., Della Morte et al '05]





Determination of the Λ-parameter

- The formula

$$\Lambda = \mu\,(b_0\bar{g}^2)^{-b_1/2b_0^2}\,\exp\left\{-\frac{1}{2b_0\bar{g}^2}\right\} \times \exp\left\{-\int_0^{\bar{g}} \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

  holds for any value of μ. We may use it at $L_{\rm min}$ to obtain

$$\Lambda L_{\rm min} = f\left(\bar{g}(L_{\rm min})\right)$$

- The function $f(g)$ can be evaluated at $g = \bar{g}(L_{\rm min})$ deep in the perturbative region. The integral in the exponent

$$\int_0^{\bar{g}} \mathrm{d}x \left[\frac{b_2 b_0 - b_1^2}{b_0^3} x + \mathrm{O}(x^3)\right] = \frac{b_2 b_0 - b_1^2}{2b_0^3}\bar{g}^2 + \mathrm{O}(\bar{g}^4)$$

  may thus be evaluated using the β-function at 3-loop order.
- Since $L_{\rm max} = 2^n L_{\rm min}$ one knows $L_{\rm max}\Lambda$
- still need $F_\pi L_{\rm max}$

Matching to a low energy scale

Ideally one would like to compute e.g. $F_\pi \Lambda$, and take $F_\pi = 132\mathrm{MeV}$ from experiment

- What is required? The scale L_max is implicitly defined:

$$\bar{g}^2(L_\mathrm{max}) = 4.84 \qquad \Rightarrow \qquad (L_\mathrm{max}/a)(g_0)$$

  Setting $L_\mathrm{max}/a = 6, 8, 10, \ldots$ one then finds corresponding values of the bare coupling (at fixed g_0 some interpolation of L_max/a will be necessary instead)

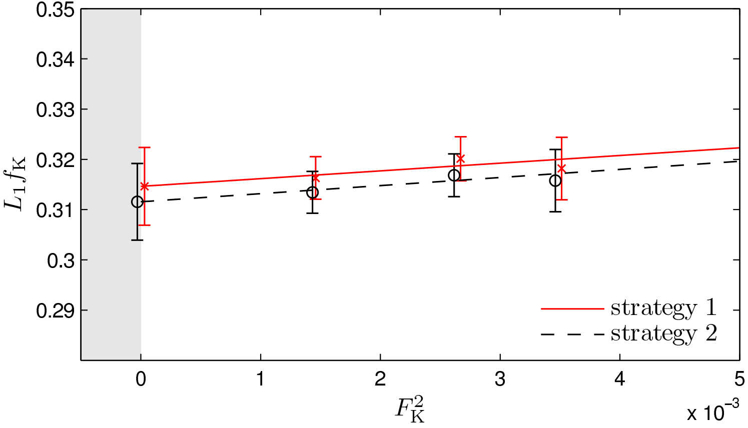
- One must then be able to compute aF_π in a large volume simulation at the very same values of the bare coupling:

$$L_\mathrm{max} F_\pi = \lim_{g_0 \to 0} (L_\mathrm{max}/a)(g_0)(aF_\pi)(g_0)$$

- One thus needs a range of g_0 where both can be computed, aF_π and $\bar{g}(L_\mathrm{max})$
- Remark: intermediate results are often quoted in terms of Sommer's scale r_0, rather than F_π.

Matching to a low energy scale

[Alpha collab. '12] Extrapolate the kaon decay constant times L_1 to the continuum (analogous $F_\pi L_{\rm max}$)



Results

- The scale r_0 [R. Sommer '93] is obtained from the force $F(r)$ between static quark and antiquark separated by a distance r:

$$r_0^2 F(r_0) = 1.65$$

The r.h.s. was chosen so that phenomenological estimates from potential models yield $r_0 = 0.5\,\mathrm{fm}$.

- Recent result for $N_\mathrm{f} = 2$ ([ALPHA '12]): $F_K = 155\,\mathrm{MeV}$ implies $r_0 = 0.503(10)\,\mathrm{fm}$ (at physical pion mass!).
- Results for Λ using $r_0 = 0.5\,\mathrm{fm}$ [ALPHA '99-'12]

$$\Lambda^{(2)}_{\overline{\mathrm{MS}}} r_0 = 0.789(52), \qquad \Lambda^{(2)}_{\overline{\mathrm{MS}}} = 310(20)\,\mathrm{MeV}$$
$$\Lambda^{(0)}_{\overline{\mathrm{MS}}} r_0 = 0.602(48), \qquad \Lambda^{(0)}_{\overline{\mathrm{MS}}} = 238(19)\,\mathrm{MeV}$$

The running quark mass

- Coupled evolution of the running mass and the coupling:

$$\begin{aligned} \overline{m}(2L) &= \sigma_m(u)\overline{m}(L), \qquad \sigma_m(u) = 1/\sigma_{\mathrm{P}} \\ \bar{g}^2(2L) &= \sigma(u) \end{aligned}$$

- Once the running coupling is known in a range $[u_0, u_n]$,

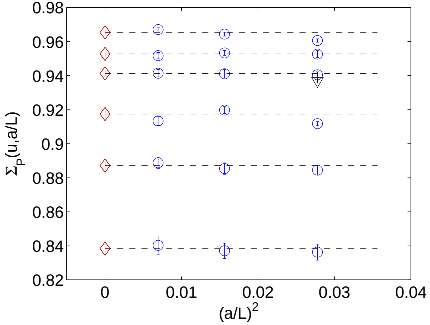
$$u_0 = \bar{g}^2(L_{\mathrm{min}}), \quad u_k = \bar{g}^2(2^k L_{\mathrm{min}}), k = 1, 2, \ldots, n$$

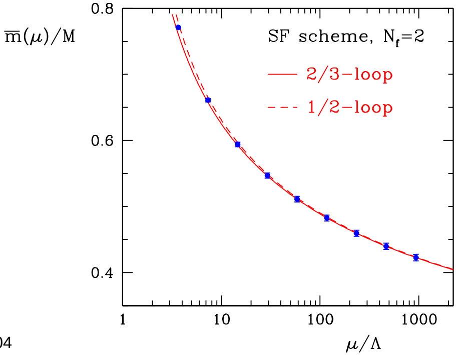
  determine $\sigma_m(u)$ for the same range of couplings: evolution of quark mass and coupling recursively

$$\overline{m}(2^k L_{\mathrm{min}})/\overline{m}(2^{k-1} L_{\mathrm{min}}) = \sigma_m(u_k), \qquad k = 1, 2, \ldots, n$$

- one obtains $\overline{m}(2L_{\mathrm{max}})/\overline{m}(L_{\mathrm{min}})$
- Extract $\overline{m}(L_{\mathrm{min}})/M$ using PT as for Λ-parameter

Running mass in the SF scheme [ALPHA '05]





Relation to bare quark masses

- In practice with Wilson type quarks, one avoids the additive renormalisation of the bare quark mass parameter by replacing it by a *measured* bare mass $m_{\rm PCAC}$ from the (bare) PCAC relation:

$$m_{\rm PCAC} \stackrel{\rm def}{=} \frac{\langle \partial_\mu A^a_\mu(x) O \rangle}{2\langle P^a(x) O \rangle}$$

- The running quark mass is then related to $m_{\rm PCAC}$

$$\overline{m}(L) = \underbrace{Z_{\rm P}^{-1}(g_0, L/a) Z_{\rm A}(g_0)}_{\text{known factors}} \underbrace{m_{\rm PCAC}(g_0)}_{\text{measured}},$$

- Combine results,

$$M = Z_M(g_0) m_{\rm PCAC}(g_0)$$

and take the continuum limit $g_0 \to 0$.

Strange quark mass

The most recent $N_\mathrm{f} = 2$ result for the strange quark mass using this strategy ([ALPHA '12]):

$$M_s = 138(3)(1)\,\mathrm{MeV} \quad \Rightarrow \quad \overline{m}^{\overline{\mathrm{MS}}}(\mu = 2\,\mathrm{GeV}) = 102(3)(1)\,\mathrm{MeV}$$

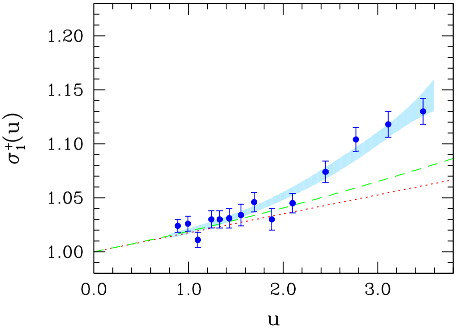
- Quoted errors are statistical and systematic;
- Note: Except for quenching of the strange quark ALL systematic errors have been addressed!

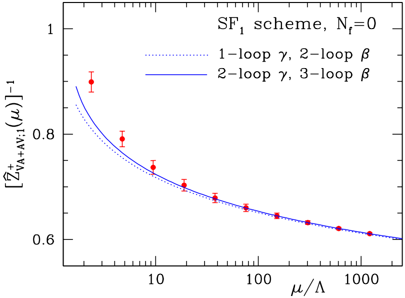
Concluding remarks

- The recursive finite volume technology has completely eliminated the problem with large scale differences. The RG running is determined in the continuum limit and universal (i.e. regularisation independent)
- To obtain physical results one needs to perform a matching calculation at a low energy scale: it is crucial to have a range in bare couplings where both, the renormalisation conditions and the hadronic input can be computed
- Whether perturbation theory for the running coupling/operator is working well or not down to low scales is not so important; you would not know this beforehand! What error estimate would you have given?!
- Many operator renormalisation problems have been treated already; the technique can be generalised to operators containing static quarks (cf. R. Sommer's Nara lectures).

Running of the B_K four-quark operator in SF scheme

Quenched approximation [ALPHA collab. '05]





Continuum vs. lattice symmetries

On the lattice symmetries are typically reduced with respect to the continuum. Examples are

1. Space-Time symmetries: the Euclidean O(4) rotations are reduced to the $O(4,\mathbb{Z})$ group of the hypercubic lattice. Other lattice geometries are possible, even random lattices have been tried.
2. Supersymmetry: only partially realisable on the lattice
3. Chiral and Flavour symmetries:
   - staggered quarks: only a $U(1)\times U(1)$ symmetry remains
   - Wilson quarks: an exact $SU(N_\mathrm{f})_\mathrm{V}$
   - twisted mass Wilson quarks: various U(1) symmetries (both axial and vector)
   - overlap/Neuberger quarks: complete continuum symmetries!
   - Domain Wall quarks: (negligibly ?) small violations of axial symmetries; consequences are analysed like for Wilson quarks

Case study: chiral and flavour symmetries with Wilson type quarks

Exact lattice Ward identities (1)

Euclidean action $S = S_{\rm f} + S_{\rm g}$:

$$S_{\rm f} = a^4 \sum_x \overline{\psi}(x)\,(D_W + m_0)\,\psi(x), \qquad S_{\rm g} = \frac{1}{g_0^2} \sum_{\mu,\nu} {\rm tr}\,\{1 - P_{\mu\nu}(x)\}$$

$$D_W = \frac{1}{2}\left\{\left(\nabla_\mu + \nabla_\mu^*\right)\gamma_\mu - a\nabla_\mu^*\nabla_\mu\right\}$$

Isospin transformations ($N_{\rm f} = 2$, $\tau^{1,2,3}$ Pauli matrices):

$$\psi(x) \rightarrow \psi'(x) = \exp\left(i\theta(x)\tfrac{1}{2}\tau^a\right)\psi(x) \approx \left(1 + \delta_{\rm V}^a(\theta)\right)\psi(x),$$

$$\overline{\psi}(x) \rightarrow \overline{\psi}'(x) = \overline{\psi}(x)\exp\left(-i\theta(x)\tfrac{1}{2}\tau^a\right)\psi(x) \approx \left(1 + \delta_{\rm V}^a(\theta)\right)\overline{\psi}(x)$$

Perform change of variables in the functional integral and expand in θ

$$\langle O[\psi,\overline{\psi},U]\rangle = Z^{-1}\int D[\psi,\overline{\psi}]D[U]{\rm e}^{-S}O[\psi,\overline{\psi},U].$$

Due to $D[\psi,\overline{\psi}] = D[\psi',\overline{\psi}']$ one finds the vector Ward identity

$$\langle\delta_{\rm V}^a(\theta)O\rangle = \langle O\delta_{\rm V}^a(\theta)S\rangle$$

Exact lattice Ward identities (2)

Variation of the action, Noether current:

$$\begin{aligned}\delta_{\rm V}^a(\theta)S &= -ia^4\sum_x \theta(x)\partial_\mu^*\widetilde{V}_\mu^a(x)\\ \widetilde{V}_\mu^a(x) &= \overline{\psi}(x)(\gamma_\mu-1)\frac{\tau^a}{4}U(x,\mu)\psi(x+a\hat{\mu})\\ &\quad +\overline{\psi}(x+a\hat{\mu})(\gamma_\mu+1)\frac{\tau^a}{4}U(x,\mu)^\dagger\psi(x)\end{aligned}$$

Choose region R and θ:

$$R=\{x:\ t_1<x_0\le t_2\},\qquad \theta(x)=\begin{cases}1 & \text{if } x\in R\\ 0 & \text{otherwise}\end{cases}$$

Exact lattice Ward identities (3)

if $O = O_{\rm ext}$ is localised outside R:

$$
\begin{aligned}
0 = \langle O_{\rm ext}\, i\delta_{\rm V}^a(\theta) S\rangle &= a^4 \sum_{x_0=t_1+a}^{t_2} \sum_{\mathbf{x}} \langle O_{\rm ext}\, \partial_\mu^* \widetilde{V}_\mu^a(x)\rangle \\
&= a \sum_{x_0=t_1+a}^{t_2} \partial_0^* \langle O_{\rm ext}\, Q_{\rm V}^a(x_0)\rangle \\
&= \langle O_{\rm ext}\, Q_{\rm V}^a(t_2)\rangle - \langle O_{\rm ext}\, Q_{\rm V}^a(t_1)\rangle
\end{aligned}
$$

i.e. the vector charge is time-independent;
This expresses the exact vector symmetry on the lattice;
N.B.: These are exact identities between *lattice* correlation functions!

Exact lattice Ward identities (4)

Choosing $O = O_{\rm ext}\widetilde{V}^b_\mu(y)$, with $y \in R$:

$$i\varepsilon^{abc}\left\langle O_{\rm ext}\widetilde{V}^c_k(y)\right\rangle = \left\langle O_{\rm ext}\widetilde{V}^b_k(y)\left[Q^a_{\rm V}(t_2) - Q^a_{\rm V}(t_1)\right]\right\rangle$$

$$i\varepsilon^{abc}\left\langle O_{\rm ext}Q^c_{\rm V}(y_0)\right\rangle = \left\langle O_{\rm ext}Q^b_{\rm V}(y_0)\left[Q^a_{\rm V}(t_2) - Q^a_{\rm V}(t_1)\right]\right\rangle$$

- N.B. The RHS does not vanish since the time ordering matters: $t_1 < y_0$ and $t_2 > y_0$
- Constitutes Euclidean version of charge algebra!

Exact lattice Ward identities (5)

- implies that the Noether current $\widetilde{V}_\mu^a$ is protected against renormalisation; if we admit a renormalisation constant $Z_{\tilde{\mathrm{V}}}$ it follows that $Z_{\tilde{\mathrm{V}}}^2 = Z_{\tilde{\mathrm{V}}}$ hence $Z_{\tilde{\mathrm{V}}} = 1$; its anomalous dimension vanishes!
- Any other definition of a lattice current, e.g. the local current

$$V_\mu^a(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x), \qquad (V_{\mathrm{R}})_\mu^a = Z_{\mathrm{V}} V_\mu^a$$

can be renormalised by comparing with the conserved current. Its anomalous dimension must vanish, i.e.

$$Z_{\mathrm{V}} = Z_{\mathrm{V}}(g_0) \quad \overset{g_0\to 0}{\sim} \quad 1 + \sum_{n=1}^{\infty} Z_{\mathrm{V}}^{(n)} g_0^{2n}.$$

Continuum chiral WI's as normalisation conditions

- For chiral symmetry there is no conserved current with Wilson quarks.
- However: expect that chiral symmetry can be restored in the continuum limit!

$\Rightarrow$ [Bochicchio et al '85]: use continuum chiral Ward identities and impose them as normalisation condition at finite lattice spacing a!

Continuum chiral WI's as normalisation conditions

- Define chiral variations:

$$\delta_{\mathrm{A}}^a(\theta)\psi(x) = i\gamma_5 \tfrac{1}{2}\tau^a\theta(x)\psi(x), \qquad \delta_{\mathrm{A}}^a(\theta)\overline{\psi}(x) = \overline{\psi}(x)i\gamma_5\tfrac{1}{2}\tau^a\theta(x)$$

- Derive formal continuum Ward identities assuming that the functional integral can be treated like an ordinary integral:

$$\Rightarrow \qquad \langle \delta_{\mathrm{A}}^a(\theta)O\rangle = \langle O\delta_{\mathrm{A}}^a(\theta)S\rangle,$$

$$\begin{aligned}
\delta_{\mathrm{A}}^a(\theta)S &= -i\int \mathrm{d}^4x\theta(x)\left(\partial_\mu A_\mu^a(x) - 2mP^a(x)\right) \\
A_\mu^a(x) &= \overline{\psi}(x)\gamma_\mu\gamma_5\tfrac{1}{2}\tau^a\psi(x), \qquad P^a(x) = \overline{\psi}(x)\gamma_5\tfrac{1}{2}\tau^a\psi(x)
\end{aligned}$$

Simplest chiral WI: the PCAC relation (1)

- Shrink the region R to a point x:

$$\begin{aligned} \langle O_{\rm ext}\delta_{\rm A}^{a}(\theta)S\rangle &= 0 \\ \Rightarrow \quad \left\langle \partial_\mu A_\mu^a(x) O_{\rm ext}\right\rangle &= 2m\left\langle P^a(x)O_{\rm ext}\right\rangle \end{aligned}$$

- The PCAC relation implies that chiral symmetry is restored in the chiral limit.

Simplest chiral WI: the PCAC relation (2)

- Impose PCAC on Wilson quarks at fixed a: define a bare PCAC mass:

$$m = \frac{\langle \partial_\mu A_\mu^a(x) O_{\rm ext} \rangle}{\langle P^a(x) O_{\rm ext} \rangle}$$

- A renormalised quark mass can thus be written in two ways

$$m_{\rm R} = Z_{\rm A} Z_{\rm P}^{-1} m = Z_m(m_0 - m_{\rm cr}) \quad \Rightarrow \quad m = Z_m Z_{\rm P} Z_{\rm A}^{-1}(m_0 - m_{\rm cr})$$

⇒ The critical mass can be determined by measuring the bare PCAC mass m as a function of m_0 and extra/interpolation to $m = 0$.

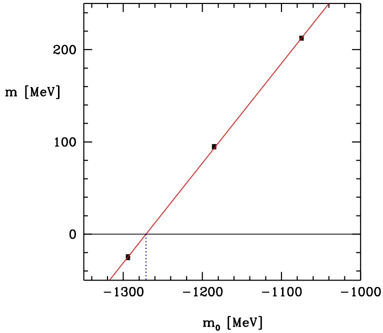
- Note: m is only defined up to O(a); any change in $O_{\rm ext}$ will lead to O(a) differences.

Determination of the critical mass

PCAC quark mass from SF correlation functions:

$$m = \frac{\partial_0 f_{\rm A}(x_0)}{2 f_{\rm P}(x_0)}$$

$8^3 \times 16$ lattice, quenched QCD, $a = 0.1\,\mathrm{fm}$



More chiral WI's: axial current normalisation

- At $m = 0$ we can derive the Euclidean current algebra (in finite volume!):

$$i\varepsilon^{abc}\left\langle O_{\rm ext}\, Q_{\rm V}^{c}(y_0)\right\rangle = \left\langle O_{\rm ext}\, Q_{\rm A}^{b}(y_0)\left[Q_{\rm A}^{a}(t_2) - Q_{\rm A}^{a}(t_1)\right]\right\rangle$$

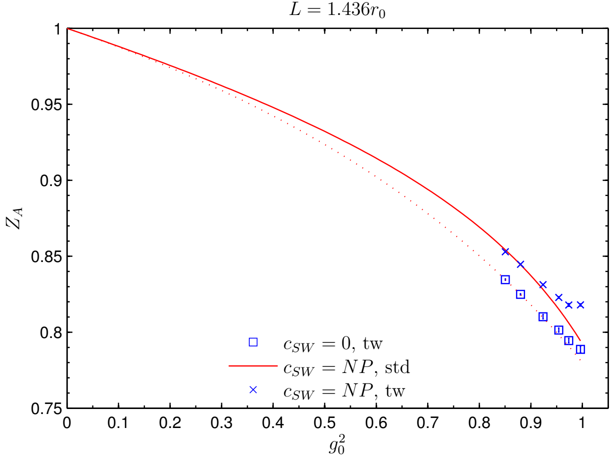
- Imposing this continuum identity on the lattice (at $m = 0$) fixes the normalisation of the axial current

$$(A_{\rm R})_{\mu}^{a} = Z_{\rm A}(g_0) A_{\mu}^{a}, \qquad Z_{\rm A}(g_0) \overset{g_0 \to 0}{\sim} 1 + \sum_{n=1}^{\infty} Z_{\rm A}^{(n)} g_0^{2n}.$$

- Note: When changing the external fields $O_{\rm ext}$, the result for $Z_{\rm A}$ will change by terms of $\mathrm{O}(a)$.
- The PCAC relation and the charge algebra become operator identities in Minkowski space. Changing $O_{\rm ext}$ corresponds to looking at different matrix elements of these operator identities. On the lattice these must be equal up to $\mathrm{O}(a)$ terms.

Axial current normalisation with Wilson quarks

$Z_{\rm A}$ in quenched approximation [Lüscher et al. '96, Leder & S '10]



Similar results for $N_{\rm f} = 2, 3$ by ALPHA collab.