Non-perturbative Renormalization of Lattice QCD

Part II/V

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Kolkata, 3 December 2013

Contents

Reminder: Symanzik's effective theory

- Expansion of renormalized lattice correlation functions in powers of a;
- reveals structure of $\mathrm{O}(a)$ effects; powers of a explicit!
- Wilson quarks: both even and odd powers of a appear;
- if remnant chiral symmetry: expect only even powers of a
- bosonic theories: only even powers of a;
- logarithmic dependence of a hidden in coefficients; in PT obtain a polynomial in $\ln a$ with degree l given by loop order;

$$P(a) \sim P(0) + \sum_{n=1}^{\infty} \sum_{k=1}^{l} c_{nk}\, a^n (\ln a)^k$$

What is known about the logarithmic dependence beyond perturbation theory?

The 2d O(n) sigma model: a test laboratory for QCD

$$S = \frac{n}{2\gamma} \sum_{x,\mu} (\partial_\mu \mathbf{s})^2, \qquad \mathbf{s} = (s_1, \ldots, s_n) \qquad \mathbf{s}^2 = 1$$

- like QCD the model has a mass gap and is asymptotically free
- many analytical tools: large n expansion, Bethe ansatz, form factor bootstrap, etc.
- efficient numerical simulations due to cluster algorithms.

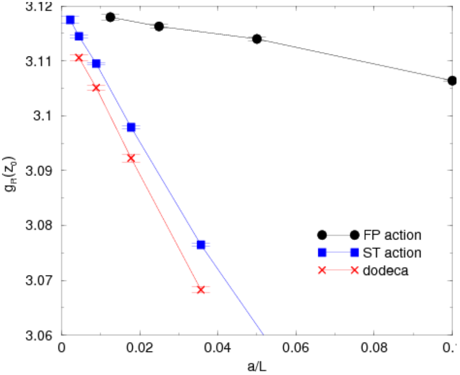
$\Rightarrow$ very precise data over a wide range of lattice spacing (a can be varied by 1-2 orders of magnitude).

- Symanzik: expect O(a^2) effects, up to logarithms
- Large n, at leading [Caracciolo, Pelissetto '98] and next-to-leading [Knechtli, Leder, Wolff '05]:

$$P(a) \sim P(0) + \frac{a^2}{L^2}\left(c_1 + c_2 \ln(a/L)\right)$$

A sobering result (1):

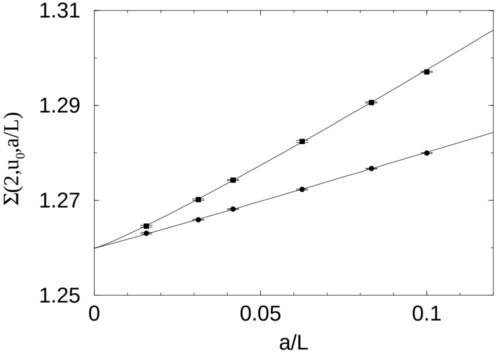
Numerical study of renormalised finite volume coupling to high precision ($n = 3$) [Hasenfratz, Niedermayer '00, Hasenbusch et al. '01, Balog et al. '09]



- Cutoff effects seem to be almost linear in a!
- Is this just an unfortunate case?

A sobering result (2):

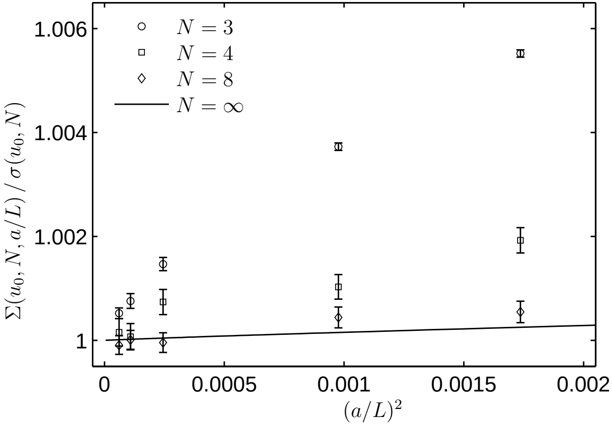
[Balog, Niedermayer & Weisz '09]



Fits with a and $a \ln a$ terms, lattice sizes $L/a = 10, \ldots, 64$

A closer look (1)

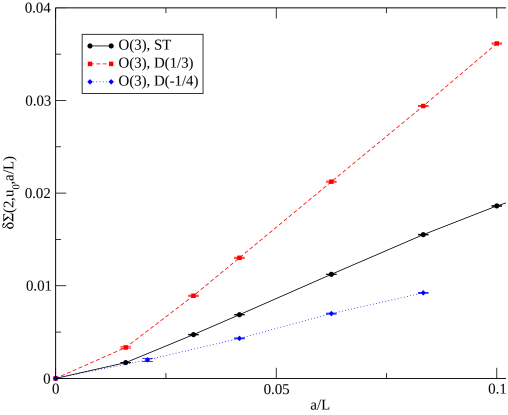
[Knechtli, Leder, Wolff '05], plot of cutoff effects vs. a^2/L^2, various n:



Asymptotic behaviour for larger n according to expectation, what about $n = 3$?

A closer look (2)

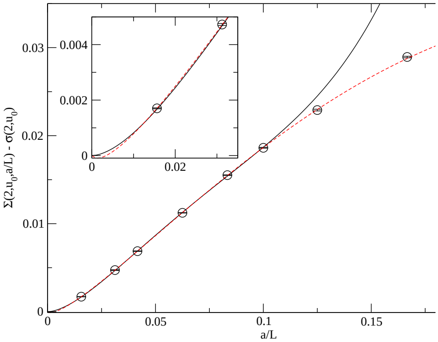
Continuum limit for mass gap $m(L)$ known analytically [Balog & Hegedus '04]! Subtract & study pure cutoff effect [Balog, Niedermayer, Weisz '09]



A closer look (3)

Continuum limit for mass gap $m(L)$ known analytically [Balog & Hegedus '04]!

Subtract & study pure cutoff effect: $\Sigma(2, u_0, a/L) - \sigma(2, u_0)$:



$c_1 a + c_2 a \ln a + c_3 a^2$ (dashed) vs. $c_1 a^2 + c_2 a^2 \ln a + c_3 a^4$ (solid)

A closer look (4) & solution of puzzle

[Balog, Niedermayer & Weisz '09]

- performed two-loop calculation with both effective Symanzik theory and lattice theory (various actions)
- Matching of both sides and subsequent RG considerations

$\Rightarrow$ Symanzik theory predicts for $\mathrm{O}(n)$ model leading $\mathrm{O}(a^2)$ behaviour:

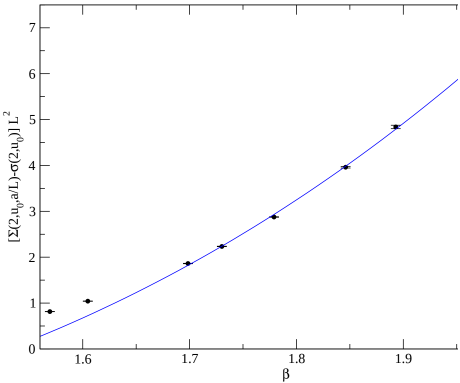
$$\delta(a) \propto a^2 \left(\ln a^2\right)^{n/(n-2)}$$

- compatible with large n result since $\lim_{n\to\infty}\{n/(n-2)\} = 1$
- For $\mathrm{O}(3)$ model:

$$\delta(a) \propto a^2 \left(\ln^3(a^2) + c_1 \ln^2(a^2) + c_2 \ln(a^2) + c_4\right) + \mathrm{O}(a^4)$$

A closer look (5)

Coefficient of $O(a^2)$ term [Balog, Niedermayer & Weisz '09]:



Not exactly constant! Multiplied with a^2 obtain "fake" linear behaviour in a!

Conclusion

- Symanzik's analysis is applicable beyond perturbation theory
- In QCD numerical results seem to confirm expectations;
- The Symanzik expansion is asymptotic, and powers of a are accompanied by (powers of) logarithms,
- Lesson from σ model: logarithmic corrections to powers in a not always negligible!
- It helps to combine results from different regularisations: renormalised quantities must agree in the continuum limit (assuming universality)

Renormalization group functions

If the renormalized coupling and quark mass are defined non-perturbatively at all scales

$\Rightarrow$ renormalization group functions are defined non-perturbatively, too:

- β-function

$$\beta(\bar{g}) = \mu \frac{\partial \bar{g}(\mu)}{\partial \mu}, \qquad \bar{g}^2(\mu) = 4\pi\alpha_{\mathrm{qq}}(1/\mu)$$

- quark mass anomalous dimension:

$$\tau(\bar{g}) = \frac{\partial \ln \overline{m}(\mu)}{\partial \ln \mu} = -\lim_{a\to 0} \left.\frac{\partial \ln Z_{\mathrm{P}}(g_0, a\mu)}{\partial \ln a\mu}\right|_{\bar{g}(\mu)}$$

Asymptotic expansion for weak couplings:

$$\begin{aligned} \beta(g) &\sim -g^3 b_0 - g^5 b_1 \ldots, \qquad & b_0 &= \left\{\tfrac{11}{3}N - \tfrac{2}{3}N_{\mathrm{f}}\right\}(4\pi)^{-2}, \ldots \\ \tau(g) &\sim -g^2 d_0 - g^4 d_1 \ldots, \qquad & d_0 &= 3(N - N^{-1})(4\pi)^{-2}, \ldots \end{aligned}$$

The Callan-Symanzik equation

Physical quantities P are independent of μ, and thus satisfy the CS-equation:

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(\bar{g})\frac{\partial}{\partial\bar{g}}+\tau(\bar{g})\overline{m}\frac{\partial}{\partial\overline{m}}\right\}P=0$$

Λ and M_i are special solutions:

$$\begin{aligned}\Lambda &= \mu\,(b_0\bar{g}^2)^{-b_1/2b_0^2}\,\exp\left\{-\frac{1}{2b_0\bar{g}^2}\right\}\\ &\quad\times\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\frac{1}{\beta(x)}+\frac{1}{b_0x^3}-\frac{b_1}{b_0^2x}\right]\right\}\\ M_i &= \overline{m}_i\,(2b_0\bar{g}^2)^{-d_0/2b_0}\exp\left\{-\int_0^{\bar{g}}\mathrm{d}x\left[\frac{\tau(x)}{\beta(x)}-\frac{d_0}{b_0x}\right]\right\}\end{aligned}$$

N.B. no approximations involved!

Λ and M_i as fundamental parameters of QCD

- defined beyond perturbation theory
- scale independent
- scheme dependence? Consider finite renormalization:

$$g'_{\mathrm{R}} = g_{\mathrm{R}} c_g(g_{\mathrm{R}}), \qquad m'_{\mathrm{R},i} = m_{\mathrm{R},i} c_m(g_{\mathrm{R}})$$

with asymptotic behaviour $c(g) \sim 1 + c^{(1)} g^2 + \ldots$
$\Rightarrow$ find the exact relations

$$M'_i = M_i, \qquad \Lambda' = \Lambda \exp(c_g^{(1)}/b_0).$$

$\Rightarrow$ $\Lambda_{\overline{\mathrm{MS}}}$ can be defined indirectly beyond PT; to obtain Λ in any other scheme requires the one-loop matching of the respective coupling constants.

Strategy to compute Λ and M_i

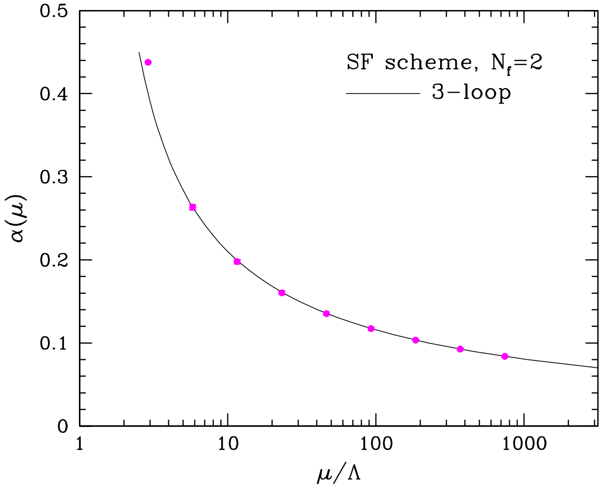
- At fixed g_0 determine the subtracted bare quark masses $m_i(g_0)$ corresponding to the chosen experimental input.
- Choose a renormalization scale μ in units of the chosen scale (e.g. F_π, r_0,...)
- Determine $\alpha_{\rm qq}(1/\mu)$ and $Z_{\rm P}(g_0, a\mu)$ at the same g_0 in the chiral limit;
- repeat the same for a range of g_0-values and take the continuum limit

$$\lim_{a\to 0} Z_{\rm P}^{-1}(g_0, a\mu) m_i(g_0), \qquad \lim_{a\to 0} \alpha_{\rm qq}(1/\mu)$$

- repeat for a range of μ values and check whether the perturbative regime has been reached;
- if this is the case, use the perturbative β- and τ-function to extrapolate to $\mu = \infty$; extract Λ and M_i (equivalently convert to $\overline{\rm MS}$ scheme deep in perturbative region).

Example: running of the coupling (SF scheme, $N_f = 2$)

[ALPHA, M. Della Morte et al. 2005]



The problem of large scale differences

Λ and M_i refer to the high energy limit of QCD

- The scale μ must reach the perturbative regime: $\mu \gg \Lambda_{\mathrm{QCD}}$
- The lattice cutoff must still be larger: $\mu \ll a^{-1}$
- The volume must be large enough to contain pions: $L \gg 1/m_\pi$
- Taken together a naive estimate gives

$$L/a \gg \mu L \gg m_\pi L \gg 1 \quad \Rightarrow L/a \simeq \mathrm{O}(10^3)$$

$\Rightarrow$ widely different scales cannot be resolved simultaneously on a finite lattice!

In practice ...

This estimate may be a little too pessimistic:

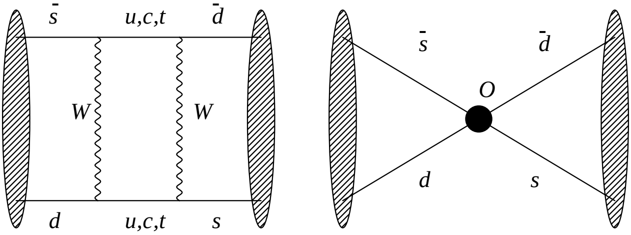
- $Lm_\pi \approx 3 - 4$ often sufficient
- if cutoff effects are quadratic one only needs $a^2\mu^2 \ll 1$.
- when working in momentum space one may argue that the cutoff really is π/a;
- in any case, one must satisfy the requirement $\mu \gg \Lambda_{\mathrm{QCD}}$

Heavy quark thresholds

Λ and M_i implicitly depend on N_f the number of active flavours! If computed for $N_\mathrm{f} = 2, 3$ one needs to perform a matching across the charm and bottom thresholds to match the real world at high energies.

QCD & composite operators (1)

Apart from the fundamental parameters of QCD one is interested in hadronic matrix elements of composite operators:
<u>Example</u>: $K^0 - \bar{K}^0$ mixing amplitude in the Standard Model:



A local interaction arises by integrating out W-bosons and t, b, c quarks, corresponding to a composite 4-quark operator

QCD & composite operators (2)

- The mixing amplitude reduces to the hadronic matrix element:

$$\begin{aligned}\langle \bar{K}^0 | O^{\Delta S=2} | K^0 \rangle &= \frac{8}{3} m_K^2 F_K^2 B_K \\ O^{\Delta S=2} &= \sum_\mu [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d]\end{aligned}$$

  $O^{\Delta S=2}$ requires a multiplicative renormalization; it is initially defined in continuum scheme used for the Operator Product Expansion (OPE)
- Other composite operators arise by applying the OPE with respect to some hard scale, such as the photon momentum in Deep Inelastic Scattering (DIS)
- We thus need to discuss renormalisation of composite operators (cf. quark mass renormalisation for a first example)

RGI operators (1)

- Consider renormalized n-point function of multiplicatively renormalizable operators O_i:

$$G_{\mathrm{R}}(x_1,\cdots,x_n;m_{\mathrm{R}},g_{\mathrm{R}})=\prod_{i=1}^{n}Z_{O_i}(g_0,a\mu)G(x_1,\cdots,x_n;m_0,g_0)$$

- Callan-Symanzik equation:

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(\bar g)\frac{\partial}{\partial\bar g}+\tau(\bar g)\overline{m}\frac{\partial}{\partial\overline{m}}+\sum_{i=1}^{n}\gamma_{O_i}(\bar g)\right\}G_R=0$$

where

$$\gamma_{O_i}(\bar g(\mu))=\left.\frac{\partial\ln Z_O(g_0,a\mu)}{\partial\ln(a\mu)}\right|_{\bar g(\mu)}$$

- Asymptotic behaviour for small couplings:

$$\gamma_O(g)\sim -g^2\gamma_O^{(0)}-g^4\gamma_O^{(1)}+\dots$$

RGI operators (2)

RGI operators can be defined as solutions to the CS equation:

$$\left(\beta(\bar{g})\frac{\partial}{\partial\bar{g}} + \gamma_O\right) O_{\mathrm{RGI}} = 0$$

where

$$O_{\mathrm{RGI}} = O_{\mathrm{R}}(\mu)\left(\frac{\bar{g}^2(\mu)}{4\pi}\right)^{-\gamma_O^{(0)}/2b_0} \exp\left\{-\int_0^{\bar{g}} \mathrm{d}x \left[\frac{\gamma_O(x)}{\beta(x)} - \frac{\gamma_O^{(0)}}{b_0 x}\right]\right\}$$

- Its name derives from the fact that O_{RGI} is renormalisation scheme independent (analogous to M_i, verify it!)!
- Beware: the overall normalisation for O_{RGI} here follows the standard convention used for B_K, which differs from the one used for M.

Perturbative vs. non-perturbative renormalisation

Distinguish 3 cases:

1. finite renormalisations: e.g. axial current normalisation for Wilson quarks $Z_A(g_0^2)$ (cf. lecture 4)

$\Rightarrow$ perturbation theory to higher orders in g_0^2 might be an option [Di Renzo et al. '2006 ff]

2. multiplicative scale dependent renormalisations, e,g, $O^{\Delta S=2}$:

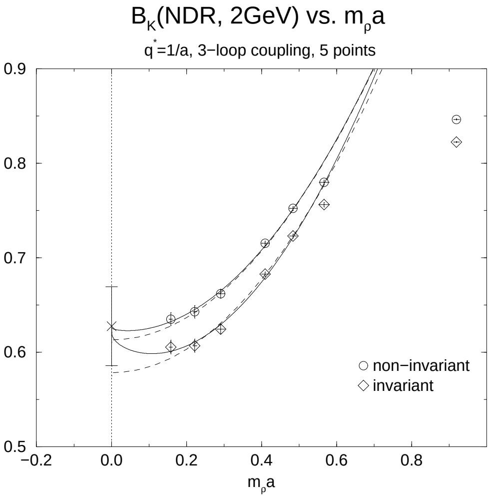
$\Rightarrow$ strong case for non-perturbative renormalisation (see below)

3. Power divergences: mixing with operators with lower dimensions, additive quark mass renormalisation with Wilson quarks:

$\Rightarrow$ total failure of perturbation theory (s. below)

Quenched B_K with staggered quarks [JLQCD, '98]

2 different discretised operators, perturbative 1-loop renormalisation



⇒ Results finite but continuum extrapolation difficult due to

Power divergences and perturbation theory

What problems arise if we just use perturbation theory?
In the case of power divergent subtraction PT is clearly insufficient:

additive mass subtraction with Wilson quarks

$$m_{\mathrm{R}} = Z_m(m_0 - m_{\mathrm{cr}}), \qquad m_{\mathrm{cr}} = \frac{1}{a} f(g_0^2)$$

Suppose one uses a perturbative expansion of f up to g_0^{2n}:

$$\Delta f(g_0^2) = \mathrm{O}(g_0^{2n}), \qquad g_0^{2n} \quad \sim \quad \frac{1}{(\ln a\Lambda)^n}$$

Remainder (after perturbative subtraction at finite order),

$$\frac{1}{a} \Delta f(g_0^2) \quad \sim \quad \{a(\ln a\Lambda)^n\}^{-1} \to \infty$$

is still divergent!

Momentum Subtraction Schemes (MOM)

Recall procedure in continuum perturbation theory:

- example: renormalisation of the pseudoscalar density $P^a(x) = \overline{\psi}(x)\gamma_5 \frac{1}{2}\tau^a \psi(x)$:
- Correlation functions in momentum space with external quark states:

$$
\begin{aligned}
\left\langle \tilde{\psi}(p)\tilde{\overline{\psi}}(q) \right\rangle &= (2\pi)^4 \delta(p+q) S(p) \qquad \text{quark propagator} \\
\left\langle \tilde{\psi}(p)\tilde{P}^a(q)\tilde{\overline{\psi}}(p') \right\rangle &= (2\pi)^4 \delta(p+q+p') S(p) \Gamma_P^a(p,q) S(p+q)\,,
\end{aligned}
$$

- At tree-level:

$$
\begin{aligned}
\Gamma_P^a(p,q)|_{\rm tree} &= \gamma_5 \tfrac{1}{2}\tau^a, \\
\Rightarrow \quad \tfrac{1}{4}\sum_{b=1}^{3} \mathrm{tr}\left\{ \gamma_5 \tau^b \Gamma_P^a(p,q)|_{\rm tree} \right\} &= 1
\end{aligned}
$$

- Renormalised fields:

$$\psi_{\mathrm{R}} = Z_\psi \psi, \qquad \overline{\psi}_{\mathrm{R}} = Z_\psi \overline{\psi}, \qquad P^a_{\mathrm{R}} = Z_{\mathrm{P}} P^a$$

$\Rightarrow$ renormalised vertex function:

$$\Gamma^a_{P,\mathrm{R}}(p,q) = Z_{\mathrm{P}} Z_\psi^{-2} \Gamma^a_P(p,q)$$

- typical MOM renormalisation condition (quark masses set to zero):

$$\Gamma^a_{P,\mathrm{R}}(p,0)|_{\mu^2=p^2} = \gamma_5 \tfrac{1}{2}\tau^a \qquad \Rightarrow \qquad Z_{\mathrm{P}} Z_\psi^{-2}$$

- equivalently using "projector":

$$\tfrac{1}{4}\sum_{b=1}^{3} \mathrm{tr}\left\{\gamma_5 \tau^b\, \Gamma^a_{P,\mathrm{R}}(p,0)|_{\mu^2=p^2}\right\} = 1$$

- Determine Z_ψ either from propagator or use MOM scheme for vertex function of a conserved current

$$\Gamma_{V,\mathrm{R}}(p,q) = Z_\psi^{-2} \Gamma_V(p,q)$$

Summary: MOM schemes in the continuum

- Renormalisation condions are imposed on vertex functions in the gauge fixed theory with external quark, gluon or ghost lines
- The vertex functions are taken in momentum space.
- A particular momentum configuration is chosen, such that the vertex function becomes a function of a single momentum p; quark masses are set to zero
- MOM condition: a renormalised vertex function at subtraction scale $\mu^2 = p^2$ equals its tree-level expression
- Can also be used to define a renormalised gauge coupling: take vertex function of either the 3-gluon vertex, the quark-gluon vertex or the ghost-gluon vertex.
- Renormalisation constants depend on the chosen gauge! Need wave function renormalisation for quark, gluon and ghost fields.