Non-perturbative Renormalization of Lattice QCD

Part I/V

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Some references

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Contents

QCD and the Standard Model of particle physics

The Standard Model (SM):

- describes strong, weak and electromagnetic interactions; gauge theory SU(3)×SU(2)×U(1)
- large scale differences, for instance:

$$m_t, m_H, m_Z, m_W = \mathrm{O}(100\,\mathrm{GeV}) \qquad m_b, m_c = \mathrm{O}(1\mathrm{GeV})$$

  with light quark masses still much lighter.

⇒ SM for energies $\ll m_W$ reduces to QCD + QED + tower of effective weak interaction vertices (4-quark-operators, 6-quark operators ...).

⇒ the structure of this effective "weak hamiltonian" is obtained perturbatively e.g in $\overline{\mathrm{MS}}$ scheme.

⇒ QCD + effective 4-quark operators a priori defined in perturbative framework at high energies.

Non-perturbative definition of QCD (1)

To define QCD as a QFT beyond perturbation theory it is not enough to write down its classical Lagrangian:

$$\mathcal{L}_{\rm QCD}(x) = \frac{1}{2g^2}\,\mathrm{tr}\,\{F_{\mu\nu}(x)F_{\mu\nu}(x)\} + \sum_{i=1}^{N_{\rm f}} \overline{\psi}_i(x)\,(\not{D} + m_i)\,\psi_i(x)$$

One needs to define the functional integral:

- Introduce a Euclidean space-time lattice and discretise the continuum action such that the doubling problem is solved
- Consider a finite space-time volume $\Rightarrow$ the functional integral becomes a finite dimensional ordinary or Grassmann integral, i.e. mathematically well defined!
- Take the infinite volume limit $L \to \infty$
- Take the continuum limit $a \to 0$

Non-perturbative definition of QCD (2)

- The infinite volume limit is reached with exponential corrections ⇒ usually not a major problem.
- Continuum limit: existence only established order by order in perturbation theory; only for selected lattice regularisations:
  - lattice QCD with Wilson quarks [Reisz '89]
  - lattice QCD with overlap/Neuberger quarks [Reisz, Rothe '99]
  - not (yet ?) for lattice QCD with staggered quarks [cf. Giedt '06]
- From asymptotic freedom expect

$$g_0^2 = g_0^2(a) \overset{a\to 0}{\sim} \frac{-1}{2b_0 \ln a}, \qquad b_0 = \tfrac{11N}{3} - \tfrac{2}{3}N_\mathrm{f}$$

Non-perturbative definition of QCD (3)

Working hypothesis: the perturbative picture is essentially correct:

- The continuum limit of lattice QCD exists and is obtained by taking $g_0 \to 0$
- Hence, QCD is asymptotically free, naive dimensional analysis applies: Non-perturbative renormalisation of QCD is based on the very same counterterm structure as in perturbation theory!
- Absence of analytical methods: try to take the continuum limit numerically, i.e. by numerical simulations of lattice QCD at decreasing values of g_0.

WARNING:

Perturbation Theory might be misleading (cp. triviality of ϕ_4^4-theory)

Renormalisation of QCD

- The basic parameters of QCD are g_0 and $m_i, \quad i = u, d, \ldots$.
- To renormalise QCD one must impose a corresponding number of renormalisation conditions
- If we only consider gauge invariant observables

$\Rightarrow$ no need to renormalize quark, gluon, ghost field and gauge parameter.

- All physical information (particle masses and energies, particle interactions) is contained in the (Euclidean) correlation functions of gauge invariant composite, local fields $\phi_i(x)$

$$\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle$$

- a priori each ϕ_i requires renormalisation, and thus further renormalisation conditions.

What would we like to achieve?

Natural question to ask:

What are the values of the fundamental parameters of QCD (and thus of the Standard Model!),

$$\alpha_s,\ m_u \approx m_d,\ m_s, \ldots$$

if we renormalise QCD in a hadronic renormalization scheme, i.e. by choosing the same number of experimentally well-measured hadron properties: $F_\pi,\ m_\pi, m_K, \ldots.$?

- QCD is regarded as a low energy approximation to the Standard Model; e.m. effects/isospin breaking effects are small ($\alpha_{e.m.} = 1/137$) and must be subtracted from experimental results.
- conceptually clean, natural question for lattice QCD
- alternative: combination of perturbation theory + assumptions ("quark hadron duality", sum rules, hadronisation Monte-Carlo, ...).

Renormalization of QCD in hadronic scheme

Sketch of the procedure, using e.g. hadronic observables F_π, m_π, m_K, m_D:

1. Choose a value of the bare coupling $g_0^2 = 6/\beta$; this determines the lattice spacing (i.e. mass independent scheme); choose some intial values for the bare quark mass parameters and a spatial lattice volume $(L/a)^3$ that is large enough to contain the hadrons ($\Rightarrow$ constraint for choice of g_0 or β in 1.);
2. tune the bare quark mass parameters such that m_π/F_π, m_K/F_π, m_D/F_π take their desired values (e.g. experimental ones, but not necessarily!)
3. The lattice spacing is obtained from $a(\beta) = (aF_\pi)(\beta)/F_\pi|_{\rm exp.}$
4. Reduce the value of g_0^2 (i.e. increase β) and increase L/a accordingly.
5. Repeat steps 1 – 4 until you run out of resources...

Auxiliary scale parameters r_0, t_0, w_0

For technical reasons one often introduces an auxiliary scale parameter:

- serves as a yardstick for precise tuning or scaling studies;
- should be easily computable (in any case easier than say F_π);
- should have a mild dependence on the quark masses;
- Example: Sommer's scale r_0 obtained from the force $F(r)$ between static quarks:

$$r_0^2 F(r_0) = 1.65 \qquad \Rightarrow r_0 \approx 0.5\,\mathrm{fm}$$

- Idea: at finite a use r_0/a rather than aF_π but also determine $r_0 F_\pi(\beta)$; Conversion to physical units from F_π is then postponed to the continuum limit.
- Advantage: constant physics conditions can be satisfied more precisely.

Future: a very convenient scale t_0 is based on the gradient flow [M. Lüscher '10]; (later also a variant w_0 by the BMW coll.)

From bare to renormalised parameters

- For g_0^2 (or β) in some interval one obtains:

$$F_\pi, m_\pi, m_K, m_D \quad \Rightarrow \quad g_0, am_{0,l}(g_0), am_{0,s}(g_0), am_{0,c}(g_0)$$

- These are bare parameters, their continuum limit vanishes!
- N.B.: due to quark confinement there is no natural definition of "physical" quark masses or the coupling constant from particle masses or interactions
- At high energy scales, $\mu \gg m_p$, one may use perturbative schemes to define renormalised parameters (e.g. dimensional regularisation and minimal subtraction)
- How can we relate the bare lattice parameters to the renormalised ones in, say, the $\overline{\mathrm{MS}}$ scheme?
- basic idea: introduce an intermediate renormalisation scheme which can be evaluated both perturbatively and non-perturbatively.

Why not use perturbation theory directly?

Shortcut: try to relate the bare parameters directly to $\overline{\mathrm{MS}}$ scheme, e.g. coupling: Allowing for a constant $d = \mathrm{O}(1)$,

$$\alpha_{\overline{\mathrm{MS}}}(d/a) = \alpha_0(a) + c_1\alpha_0^2(a) + c_2\alpha_0^3(a) + \ldots, \qquad \alpha_0 = \frac{g_0^2}{4\pi}$$

$$\overline{m}_{\overline{\mathrm{MS}}}(d/a) = m(a)\left(1 + Z_m^{(1)}\alpha_0(a) + Z_m^{(2)}\alpha_0^2(a) + \ldots\right\}$$

Main difficulties:

- Setting $\mu \propto a^{-1}$ means that cutoff effects and renormalisation effects cannot be disentangled; any change in the scale is at the same time a change in the cutoff.
- One needs to assume that the cutoff scale d/a is in the perturbative region, higher order effects negligible.
- One furthermore assumes that cutoff effects are negligible

$\Rightarrow$ how reliable are the error estimates?

Non-perturbatively defined renormalized parameters

Example for a renormalised coupling

Consider the force $F(r)$ between static quarks at a distance r, and *define*

$$\alpha_{\rm qq}(r) = r^2 F(r)|_{m_{\rm q,\mathit{i}}=0}$$

- at short distances:

$$\alpha_{\rm qq}(r) = \alpha_{\overline{\rm MS}}(\mu) + c_1(r\mu)\alpha^2_{\overline{\rm MS}}(\mu) + \ldots$$

- at large distances:

$$\lim_{r\to\infty} \alpha_{\rm qq}(r) = \begin{cases} \infty & \text{for } N_{\rm f} = 0 \\ 0 & \text{for } N_{\rm f} > 0 \end{cases}$$

- NB: renormalization condition is imposed in the chiral limit $\Rightarrow$ $\alpha_{\rm qq}(r)$ and its β-function are quark mass independent.

Example for a renormalised quark mass

Use PCAC relation as starting point:

$$\partial_\mu (A_\mathrm{R})^a_\mu = 2m_\mathrm{R}(P_\mathrm{R})^a$$

- A^a_μ, P^a: isotriplet axial current & density
- The normalization of the axial current is fixed by current algebra (i.e. axial Ward identities) and scale independent!

$\Rightarrow$ Quark mass renormalization is inverse to the renormalization of the axial density:

$$(P_\mathrm{R})^a = Z_\mathrm{P} P^a, \qquad m_\mathrm{R} = Z_\mathrm{P}^{-1} m_\mathrm{q}.$$

$\Rightarrow$ Impose renormalization condition for the axial density rather than for the quark mass

Renormalization condition for axial density

Define $\langle P^a_{\mathrm{R}}(x) P^b_{\mathrm{R}}(y)\rangle = \delta^{ab} G_{\mathrm{PP}}(x-y)$, and impose the condition

$$G_{\mathrm{PP}}(x)\Big|_{\mu^2x^2=1,\, m_{\mathrm{q},i}=0} = -\frac{1}{2\pi^4(x^2)^3}$$

$G_{\mathrm{PP}}(x)$ is defined at all distances:

$$G_{\mathrm{PP}}(x) \overset{x^2\to 0}{\sim} -\frac{1}{2\pi^4(x^2)^3} + \mathrm{O}(g^2), \qquad G_{\mathrm{PP}}(x) \overset{x^2\to\infty}{\sim} -\frac{1}{4\pi^2 x^2} G_\pi^2 + \ldots$$

$\Rightarrow Z_{\mathrm{P}}$ is defined at all scales μ:

- at large μ (but $\mu \ll 1/a$):

$$Z_P(g_0, a\mu) = 1 + g_0^2 d_0 \ln(a\mu) + \ldots,$$

- at low scales μ:

$$Z_{\mathrm{P}}(g_0, a\mu) \propto \mu^2$$

Lattice QCD with Wilson quarks

The action $S = S_{\rm f} + S_{\rm g}$ is given by

$$S_{\rm f} = a^4 \sum_x \overline{\psi}(x)\,(D_W + m_0)\,\psi(x), \qquad S_{\rm g} = \frac{1}{g_0^2}\sum_{\mu,\nu} \mathrm{tr}\,\{1 - P_{\mu\nu}(x)\}$$

$$D_W = \frac{1}{2}\left\{\left(\nabla_\mu + \nabla_\mu^*\right)\gamma_\mu - a\nabla_\mu^*\nabla_\mu\right\}$$

- Symmetries: $\mathrm{U}(N_{\rm f})_{\rm V}$ (mass degenerate quarks), P, C, T and $O(4,\mathbb{Z})$

$\Rightarrow$ Renormalized parameters:

$$g_{\rm R}^2 = Z_g g_0^2, \qquad m_{\rm R} = Z_m\,(m_0 - m_{\rm cr})\,, \qquad am_{\rm cr} = am_{\rm cr}(g_0).$$

- In general: $Z = Z(g_0^2, a\mu, am_0)$;
- Quark mass independent renormalisation schemes: $Z = Z(g_0^2, a\mu)$
- Simple non-singlet composite fields, e.g. $P^a = \overline{\psi}\gamma_5\tau^a\psi$ renormalise multiplicatively, $P_{\rm R}^a = Z_{\rm P}(g_0^2, a\mu, am_0)P^a$

Approach to the continuum limit (1)

Suppose we have renormalised lattice QCD non-perturbatively, how is the the continuuum limit approached?
Symanzik's effective continuum theory [Symanzik '79]:

- purpose: render the a-dependence of lattice correlation functions explicit. $\Rightarrow$ structural insight into the nature of cutoff effects
- at scales far below the cutoff a^{-1}, the lattice theory is effectively continuum like; the influence of cutoff effects is expanded in powers of a:

$$S_{\mathrm{eff}} = S_0 + aS_1 + a^2 S_2 + \dots, \qquad S_0 = S_{\mathrm{QCD}}^{\mathrm{cont}}$$

$$S_k = \int \mathrm{d}^4 x \, \mathcal{L}_{\mathrm{k}}(x)$$

$\mathcal{L}_{\mathrm{k}}(x)$: linear combination of fields

- with canonical dimension $4 + k$
- which share all the symmetries with the lattice action

Approach to the continuum limit (2)

A complete set of dimension 5 fields for $\mathcal{L}_1$ is given by:

$$\overline{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \qquad \overline{\psi}D_\mu D_\mu\psi, \qquad m\overline{\psi}\not{D}\psi, \quad m^2\overline{\psi}\psi, \qquad m\,\mathrm{tr}\{F_{\mu\nu}F_{\mu\nu}\}$$

The same procedure applies to composite fields:

$$\phi_{\mathrm{eff}}(x) = \phi_0 + a\phi_1 + a^2\phi_2 \ldots$$

for instance: $\phi(x) = P^a(x)$, basis for ϕ_1:

$$m\overline{\psi}\gamma_5\tfrac{1}{2}\tau^a\psi, \qquad \overline{\psi}\overleftarrow{\not{D}}\gamma_5\tfrac{1}{2}\tau^a\psi - \overline{\psi}\gamma_5\tfrac{1}{2}\tau^a\not{D}\psi$$

Consider renormalised, connected lattice n-point functions of a multiplicatively renormalisable field ϕ

$$G_n(x_1,\ldots,x_n) = Z_\phi^n\langle\phi(x_1)\cdots\phi(x_n)\rangle_{\mathrm{con}}$$

Approach to the continuum limit (3)

Effective field theory description:

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= \langle\phi_0(x_1)\ldots\phi_0(x_n)\rangle_{\rm con} \\ &\quad + a\int \mathrm{d}^4y\ \langle\phi_0(x_1)\ldots\phi_0(x_n)\mathcal{L}_1(y)\rangle_{\rm con} \\ &\quad + a\sum_{k=1}^{n}\langle\phi_0(x_1)\ldots\phi_1(x_k)\ldots\phi_0(x_n)\rangle_{\rm con} + \mathrm{O}(a^2) \end{aligned}$$

- $\langle\cdots\rangle$ is defined w.r.t. continuum theory with S_0
- the a-dependence is now explicit, up to logarithms, which are hidden in the coefficients.
- In perturbation theory one expects at l-loop order:

$$P(a) \sim P(0) + \sum_{n=1}^{\infty}\sum_{k=1}^{l} c_{nk}\, a^n (\ln a)^k$$

where e.g. $P(a) = G_n$ at fixed arguments.

Approach to the continuum limit (4)

Conclusions from Symanzik's analysis:

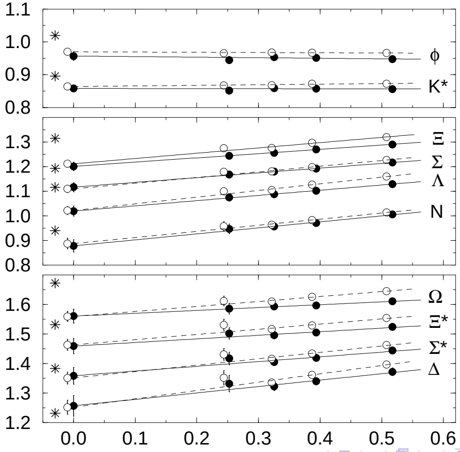
- Asymptotically, cutoff effects are powers in a, modified by logarithms;
- In contrast to Wilson quarks, only even powers of a are expected for
  - bosonic theories (e.g. pure gauge theories, scalar field theories)
  - fermionic theories which retain a remnant axial symmetry (overlap, Domain Wall Quarks, staggered quarks, Wilson quarks with a twisted mass term, etc.)

In QCD simulations a is typically varied by a factor 2

$\Rightarrow$ logarithms vary too slowly to be resolved; linear or quadratic fits (in a resp. a^2) are used in practice.

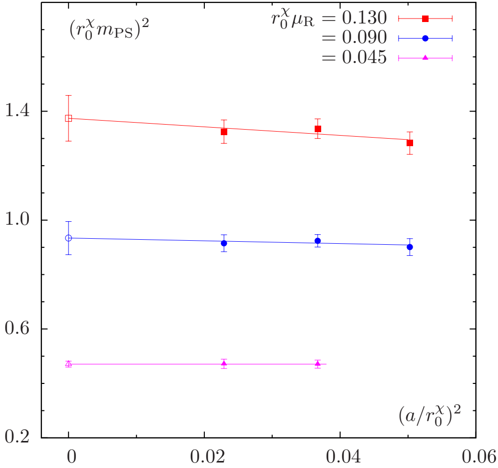
Example 1: quenched hadron spectrum

Linear continuum extrapolation of the quenched hadron spectrum; standard Wilson quarks with Wilson's plaquette action:[CP-PACS coll., Aoki et al. '02] $a = 0.05 - 0.1\,\mathrm{fm}$, experimental input: $m_K,\ m_\pi,\ m_\rho$



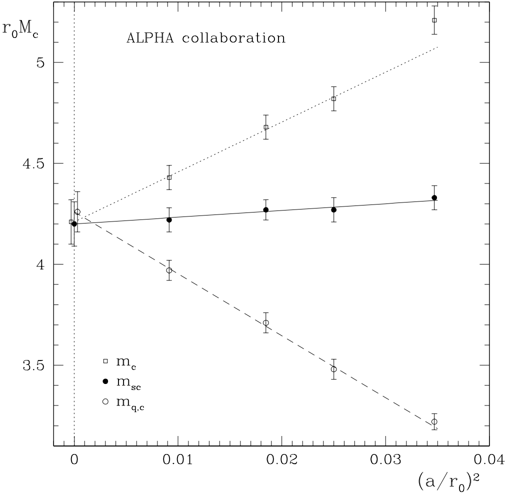
Example 2: pion mass in $N_f = 2$ tmQCD

[ETM coll. Baron et al '09]



Example 3: O(a) improved charm quark mass (quenched)

[ALPHA coll. J. Rolf et al '02]



Example 3: Step Scaling Function for SF coupling ($N_f = 2$)

[ALPHA coll., Della Morte et al. 2005]