Nonlinear Integrable Systems: Classical and Quantum

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Major Directions of My Research :

• Quantum Integrable Systems [1991-present]

1. Anjan Kundu, *Phys. Rev. Lett.* 82 (1999) 3936 : Algebraic approach in unifying quantum integrable models
2. Anjan Kundu, *Phys. Rev. Lett.*, 83 (1999) 1275 : Exact solution of double-delta function Bose gas through interacting anyon gas
3. L Hlavaty & Anjan Kundu, *Int J. Mod. Phys.* A 11 (1996) 2143-2165 : Quantum Integrability of Nonultralocal Models through Baxterisation of Quantised Braided algebra

• Classical Integrable Systems [1983-present]

1. S. Ghosh & Anjan Kundu *Phys. Rev. Lett.* 63, (1989) 1207-10 : Test of integrability for SU(2) nonlinear σ- models
2. Anjan Kundu *J. Math. Phys.* 25, (l984) 3433-8 : Landau-Lifshitz and higher order nonlinear systems gauge generated from NLS type equations

• Field Models with Topological Charge [1979-1982; Revived Presently]

1. Anjan Kundu & Yu. Rybakov *J. Phys.* A 25, (l983) 269-75: Closed vortex-type solution with Hopf index
2. Anjan Kundu *Phys. Lett.* B 110, (1982) 61-3 : Instanton in anisotropic sigma-models

Recent Publications (Review period: 2007-2010)

2007:

1. Anjan Kundu, *Phys. Rev. Lett.* 99 (2007) 154101-4 : Shape changing and accelerating solitons in integrable variable mass sine-Gordon model

2. Anjan Kundu, *Symmetry Integrability Geometry Methods & Applications* (SIGMA) 3 (2007) 040, 14 pages : q-boson in quantum integrable systems
(*Invited Review for Vadim Kuznetsov memorial volume*)

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2008:

4. Anjan Kundu, *J. Phys.* A 41 (2008) 495201 : Exact accelerating soliton in nonholonomic deformation of the KdV equation with two-fold integrable hierarchy.

5. M. Batchelor, X. Guan & Anjan Kundu, *J. Phys.* A 41 (2008) 352002 [Fast Track Communication (FTC)] : D-anyons: one-dimensional anyons with competing δ-function and derivative δ-function potentials
(*EDITOR's Choice: as best papers in FTC)*

6. I. Habibullin & Anjan Kundu , *Nucl. Phys.* B [FS], 795 (2008) 549-568 : Quantum and classical integrable sine-Gordon model with defect

7. Anjan Kundu, *J. Nonlin. Math. Phys.* 15 (3) (2008) 227-240 : Changing Solitons in Classical & Quantum Integrable Defect and Variable Mass Sine-Gordon Model

2009:

8. Anjan Kundu, R. Sahadevan & L. Nalinidevi, *J. Phys.* A 42 (2009) 115213 : Nonholonomic deformation of KdV and mKdV equations and their symmetries, hierarchies and integrability

9. Anjan Kundu, *Phys. Rev* E 79 (2009) 015601(R) [Rapid Communications] : Integrable nonautonomous NLS equations are equivalent to the standard NLS

10. Anjan Kundu, *J. Math Phys.* 50 (2009) 102702 : Nonlinearizing linear equations to integrable systems including new hierarchies of nonholonomic deformations

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11. Anjan Kundu, *SIGMA* 6 (2010), 080, 9 pages : Quantum integrable 1D anyon models: construction through braided Yang-Baxter equation, ArXiv/1005.4603

12. Anjan Kundu, *J. Math Phys.* 51 (2010) 022901 : Two-fold integrable hierarchy of nonholonomic deformation of the DNLS and the Lenells-Fokas equation

Recent:(in process)

13. Anjan Kundu, arXiv:1009.2641, (sub. to PRL) : Hidden possibilities in controlling optical soliton in fiber guided doped resonant medium

14. Anjan Kundu, arXiv: 0906.3188 : Exact N-Skyrmions with noncircular symmetry in helical magnets without inversion symmetry

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I. Quantum Integrable Systems

Unlike Boson & Fermion, Anyon live in 2D (space)

However in 1D:

i) *Anyons retain their basic properties* :

$$\Phi(x_1, x_2) = e^{-i\theta}\Phi(x_2, x_1)$$

Note: $\theta = 0$, Bosonic (*Symmetric*), $\theta = \pi$, Fermionic (*Anti-Symmetric*)

ii) *Can be constructed as Integrable models*

(Experiment. verification of 1D anyon : Keilmann et al arXiv 1009.2036 (2010))

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Known Exactly (Bethe-ansatz) Solvable Models:

1. δ- function Bose gas [Lieb-Liniger,1963]

$$\mathbf{H_N^{(1)} = -\sum_k^N \partial_{x_k}^2 + \sum_{<k,l>} V(x_k - x_l), \quad V(x_k - x_l) = c\delta(x_k - x_l)}$$

2. Derivative δ- function Bose gas (with potential $V(x_k - x_l) = c\partial_{x_k}\delta(x_k - x_l)$ (Snirman etal, 1994)

3. Attempts to build *double*-δ- function Bose gas (with $V(x_k - x_l) \sim \delta^2(x_k - x_l)$), But Failed!

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But Failed!

4. We proposed **Exactly solvable δ-function Anyon gas**
(A Kundu, PRL 1999)

Equivalent to Double δ- Bose gas!

Subsequently we also proposed:

Exactly solvable derivative-δ-function Anyon gas [Batchelor-Guan-A Kundu, JPA (2008)]

Note: *Contrary to existing belief Anyon models (without (anti-)symmetry) were Bethe-ansatz solvable*!!

Presently 1D - Anyon models became much popular:

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" ..We consider an integrable model of anyon... *introduced and solved by Kundu* [10] ...Hamiltonian [10]..." (cited 5 times)

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2). Phys. Rev. Lett. **97** 100402 (2006), *M D Girareau*:
" *Kundu [28] defines anyon field operators* $\psi_A(x)$ in terms of Bose operators ....which is *similar to Kundu's* E_g[28]..." (cited 4 times)

3). J. Phys. **40** 14963 (2007), *V Korepin et al*:
" .. the model was *introduced by Kundu [13] who also provided the Bethe-ansatz solution.*

4). arXiv:0712.1264, *R G Zhu et al*:
" .. *In a seminal work Kundu [10]* defined a 1D field operator ...the plus sign for the *Kundu's anyons* ...1D *anyon models for Kundu's* .... " (cited 6 times)

5). PhD Thesis (Stony Brook, 2009), *I O Pâtu*:

" From the historical point of view, the *first* continuous model of *1D anyons was introduce by Kundu... "*

Outstanding Problems:

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1. Known Anyon gas models still behave like bosons/fermions at the coinciding points.

Question: *Can we construct Anyon models with anyonic CR at all points ?*

2. δ and $\delta^{'}$ function Bose gases are N-particle sectors of **Quantum Integrable Field models** like *Nonlinear Schödinger equation (NLS)*: $H^{(b)} = \int dx(\psi_x^{\dagger}\psi_x + c(\psi^{\dagger}\psi)^2)$

and *Derivative NLS*:

in bosonic field :$[\psi(x), \psi^{\dagger}(y)] = \delta(x-y)$

• Questions : i) Are there novel integrable *Anyonic QFT models* corresponding to known solvable Anyon gases (proposed by us) ?

ii) How to construct such *Anyonic Lattice & Field models* in a systematic way through Yang-Baxter Equation?

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- Way out is: *Braided* YBE (BYBE) developed for solving *Non-Ultralocal quantum systems*

(Anjan Kundu & V Hlavaty , IJMPA, 1996)
BYBE:

$$R(u-v)Z^{-1}L_j(u)ZL_j(v) = Z^{-1}L_j(v)ZL_j(u)R(u-v),$$

where R-matrix, Z-braiding matrix, $L_{1j}(u)$-Lax operator at lattice site $j = 1, 2, \ldots .N$
together with braiding relation (BR) :

$$L_k(v)Z^{-1}L_j(u) = Z^{-1}L_j(u)ZL_k(v)Z^{-1}$$

for $k > j$, representing nonultralocality
For Anyon take $Z = diag(1, 1, 1, e^{i\theta})$!)
Note: for standard YBE $Z = 1$ with BR $--\,>$ commuting $[L_k(v), L_j(u)] = 0$ (ultralocal!).

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• **Novel Quantum Integrable & Bethe ansatz solvable Lattice and Field Anyon Models**
(A Kundu, SIGMA' 2010)
Using rational R-matrix we obtain:
• i) *Next-nearest neighbor interacting lattice Anyon model*

$$\mathrm{H^{a1}} = \sum_{\mathrm{k}} (\tilde{\psi}_{\mathrm{k+1}}\psi_{\mathrm{k-1}} - (\mathbf{n_k} + \mathbf{n_{k+1}})\tilde{\psi}_{\mathrm{k+1}}\psi_{\mathrm{k}} + \frac{\mathbf{1}}{\mathbf{3\Delta^2}}\mathbf{n_k^3},)$$

with anyonic CR at coinciding ($k = j$) and noncoinciding points $k > j$:

$$\psi_k \tilde{\psi}_k - e^{-i\theta} \tilde{\psi}_k \psi_k = p_k \quad \psi_k \tilde{\psi}_j = e^{i\theta} \tilde{\psi}_j \psi_k$$

Note:
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- ii) *Novel NLS anyonic quantum field model*

At continuous limit :$k \rightarrow x$, field $\psi_k \rightarrow A(x)$, obtain *integrable NLS anyonic field model*

$$\boxed{\mathbf{\hat{H}^{a2} = \int dx(A_x^\dagger A_x + c(A^\dagger A)^2)}}$$

with CR: at $x \rightarrow y^+$

$$A(x)A^\dagger(y) - e^{i\theta} A^\dagger(y)A(x) = \delta(x - y),$$

at $x > y$:

$$A(x)A^\dagger(y) = e^{i\theta} A^\dagger(y)A(x), \;\; A(x)A(y) = e^{-i\theta} A(y)A(x),$$

Note:
1) Anyon CR at all points: Boson (at $\theta = 0$) and Fermion (at $\theta = \pi$)

2) N-particle sector of Anyon NLS yields δ Anyon gas, solving the outstanding problem !.
Using trigonometric R-matrix we can similarly build

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- iii) *An integrable anyonic q-oscillator model*

and at its field limit

- iv) *Novel derivative-NLS anyonic quantum field model*

$$\boxed{\mathbf{H}^{(a4)} = \int \mathbf{dx}(\mathbf{A}_x^\dagger \mathbf{A}_x + \mathbf{2i}\kappa(\mathbf{A}^\dagger)^2 \mathbf{A}\mathbf{A}_x)}$$

yielding at N-particle sector $\partial_x \delta$- Anyon gas !

Novel Anyonic Quantum Group Algebra
As a byproduct we derive a completely New

- v) *Two-parameter Deformed Nonultralocal quantum Algebra* (different from twisting!)

$$\boxed{\mathbf{S}^+\mathbf{S}^- - \mathbf{s}\mathbf{S}^+\mathbf{S}^+ = [\mathbf{2S}^3]_q \mathbf{s}^{-\mathbf{S}^3},}$$

with

$$q^{S^3} S^{\pm} = q^{\pm 1} S^{\pm} q^{S^3}, \quad s^{S^3} S^{\pm} = s^{\pm 1} S^{\pm} s^{S^3}$$

Note: This is a Novel Hopf algebra with deforming parameters q and $s = e^{i\theta}$ (Anyonic parameters) and nontrivial: (co-)multiplication

Future Problem:

i) Quantum double, Dual Hopf algebra of Anyonic Quantum Group?

ii) Anyonic Vertex model ? (a novel concept!)

iii) Bethe ansatz Solution of the Integrable Anyon models discovered ?

iv) Form-factor, Correlation function for Anyonic NLS, derivative NLS?

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II. Classical Integrable Systems

• i) *Sine-Gordon Model with Defect and Variable Mass*

o Motivation:

sine-Gordon (SG) eqn. (in light-cone cord.)

$\theta_{xt} - m^2 \sin\theta = 0, \;\; (\partial_{xt}\theta \equiv \theta_{xt})$

is an integrable system with many applications.

However in physical situations often

i) mass becomes: $m = m(x)$ due to inhomogeneity

ii) Defect points appear in space

(Example:

1) Information propagation through active promoter zone in DNA chain due to different number of lighter (AT) and heavier (GC) base pairs

2) Inhomogeneity/defect in Josephson junction etc.)

- Question: Can we include these abberations, preserving still the Integrability?

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Our Result:

i) Variable mass $m = m(x, t)$ SG model with certain condition can be *Integrable* with unusual *Shape changing and accelerating soliton* (stable localized solution)

(A. Kundu, PRL 2007)

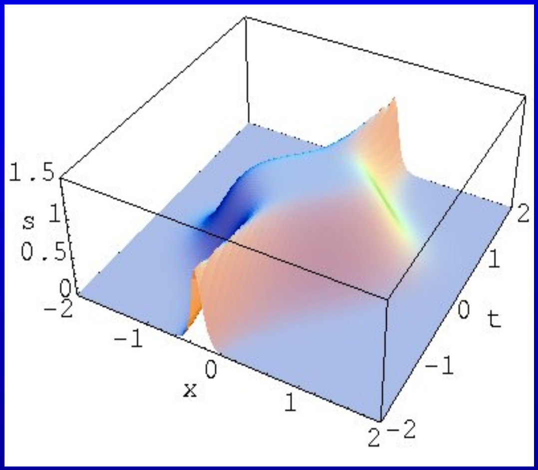

Figure 1: Soliton in Variable mass ($m = x^2 - t^2$) SG Eq. with curious change in soliton shape.

ii) SG model with a defect can be *Integrable* (Classical & Quantum) with intriguing possibility of Soliton creation/annihilation/scattering by the defect point

(Habibullin & A Kundu, Nucl.Phys.B 2008)

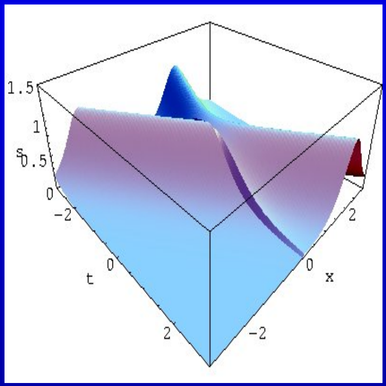

Figure 2: Creation of SG Soliton (from 1 $\longrightarrow$ 2) by the defect at center

- ii) *Nonlinearizing Linear Equations to Integrable Systems and Beyond*

o Motivation:

Nonlinear Integrable eqns. have very specific *Nonlinearity* :

i. NLS: $iq_t - q_{xx} + 2|q|^2 q = 0$, comp. conj.

ii. Derivative NLS: $iq_t - q_{xx} + 2(|q|^2 q)_x = 0$, comp. conj.

iii. KdV: $u_t - u_{xxx} + 6uu_x = 0$

iv. MKdV: $u_t - u_{xxx} + 6u^2 u_x = 0$

v. Sine-Gordon: $\theta_{xt} - \sin\theta = 0$

etc.

o Question:

Soliton exists due to balance between linear dispersive & nonlinear term

Therefore, Knowing only linear dispersive term Can we build the *Integrable Nonlinearity* ?

Going beyond Can we get New Integrable Systems by *Deforming* them ?

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- **Our Result:**

i) Integrable equations can be associated with Lax equations

$$\Phi_x = L(\lambda)\Phi, \;\; \Phi_t = V_n(\lambda)\Phi$$

with Lax-matrix pair $(L(\lambda), V_n(\lambda))$ through compatibility condition

$$L(\lambda)_t - V_n(\lambda)_x + [L(\lambda), V_n(\lambda)] = 0$$

yield the Nonlinear Eqs. (as above)

Our Scheme:

1) Starting from Linear Eq. we construct first the *naive Linear Lax-pair* and

Then identifying the *Building Blocks of the Lax-pair* and using physical notion of *Scaling Dimension* build the *Nonlinear Lax-pair* and hence get the *Nonlinear Integrable Eq.* (from compatibility condition) (all listed above!)

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Note: Our original scheme is the simplest and most physical method, for constructing Lax pair. (A Kundu, J MathPhys 2009)

2) *Deforming* the time-Lax operator V_n by additional matrix $V_{def}(g)$, with perturbing field $g(x,t)$ we can achieve *Integrable perturbation* of original Eqs. like

$$\text{Perturbed NLS}: \; iq_t - q_{xx} - 2|q|^2 q = p,$$

with the perturbing function $g(x,t)$ under *nonholonomic constraint* (i.e differential constraint)

$$p_x = -2iNq, \; N_x = i(qp^* - q^*p)$$

• We construct similar Novel *Integrable perturbation* for derivative-NLS, sine-Gordon, KdV, mKdV, 3-wave equation, Painleve II etc. All of them exhibit richer Integrability properties:

1) Exact Soliton with accelerating motion
2) Two-fold (+ve and -ve) *Integrable Hierarchy*
3) New applications with coupled systems

We demonstrate this on the representative example of

- *Integrable Perturbed NLS*

Exact accelerating Perturbed NLS soliton (with the soliton velocity: $V = V_{NLS} + V_{def}(t)$, involving an arbitrary function $c(t) = N|_{t \to -\infty}$)

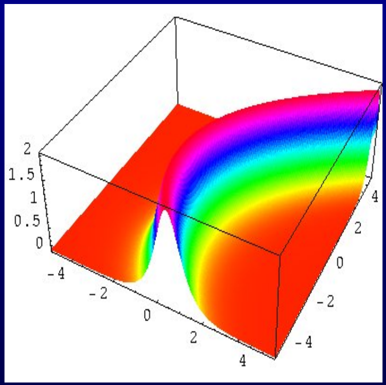

Figure 3: Exact soliton $|q(x - V(t))|$ moving with unusual constant acceleration due to t-dependent $V(t)$

• **Application to Nonlinear Fiber Optic Communication**

Note: Remarkably, Interpreting NLS field q = electrical (*optical*) field,
perturbing function p = induced *polarization* and
function N == *population inversion*
(of two level atoms of the doped fiber medium)
Perturbed NLS = model for optical communication!
with : Nonholonomic constraint $\longrightarrow$ Self-induced Transparency (SIT) Eqs.

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• Our Result:

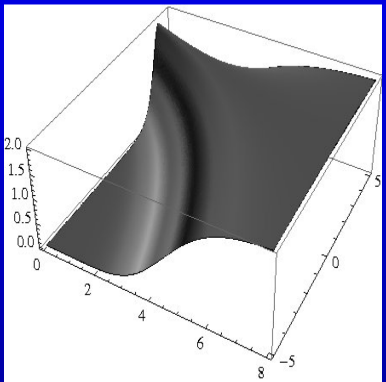
Using our insight of the theory of Integrable perturbation of NLS we could:

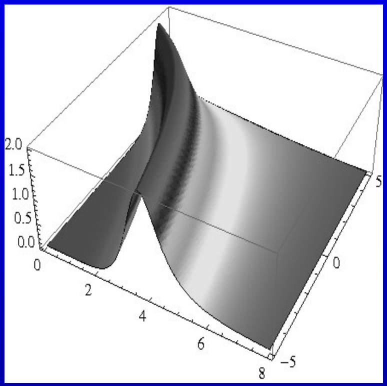
1) Discover new possibilities of *amplification* and *width control* (crucial in Nonlinear optics) of optical solitons by adjusting the arbitrary function $c(z)$ (= initial population inversion: $N|_{t\to-\infty}$)

2) Using the *hierarchy of nonholonomic constrains* we could predict more efficient soliton control by a novel proposal of coupling to *multiple SIT systems* (by multiple doping).

(A Kundu, sub. to PRL 2010)

Figure 4: a) Broadening of optical soliton $|E(z,t)|$ moving along the fiber. b) Efficient control of the soliton broadening by coupling to double SIT system, showing restoration of the soliton width and constant change in the soliton velocity.





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Note: We not only *Unify* all Integrable Eqs. in Nonlin optics (like: NLS-SIT, Hirota-SIT NLS-multiple SIT) through integrable perturbation theory but also bring similar Unifications for all other models like perturbed KdV, mKdV, sine-Gordon, derivative NLS , 3-wave eqn and even to Painleve II Eqs.

- **Future plan**

- To explore this new large class of perturbed integrable systems and
- Explore their possible Applications

• Gauge Unification of NLS class :

Idea of similar Unification stems from our earlier work, where we Unified through gauge transformation a large class of NLS Eqs:

NLS, derivative NLS, mixed NLS, Chen-Lie-Liu Eq., Gerdjikov-Ivanov Eq. , Landau-Lifshits Eq.(LLE)

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Citations (in Title of Papers):

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1). J. Phys. A **42** 465203 (2009), *Levi D. et al*: **Kundu-Eckhaus Equation** & its Discretization

2). Phys. Scripta **64** 7 (2001), *Feng ZS et al*: Explicit Exact Solitary Wave Solution for the **Kundu Equation** and the DNLS

3). arXiv:nlin/0702050 (2007), *D Levko*: Modeling of **Kundu-Eckhaus equation**

4). *Book*: Direct & Inverse Methods in Nonlinear Evolution Equations, *R Conte & A Greco*

(Science, 2003): " ... *The nonintegrable Kundu-Eckhaus equation the PDE..* "

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1). Phys. Lett. A **373** 2546 (2009), *A Biswas* : 1 soliton solution of the generalized **Radhakrishnan-Kundu-Lakshmanan equation**
2). Math. & Comp. Simul. *H Triki and Thiab Taha* **80** 849 (2009): Exact analytic solitary wave solutions for the **RKL model**
3). Chaos Solitons & Fractals **37** 215 (2008), *JL Zhang & ML Wang* : Various exact solutions for two special type **RKL models**
4) Acta Appl. Math. **104** 201 (2008) *DD Ganji etal* : Exp-Function based solution of nonlinear **Radhakrishnan-Kundu-Lakshmanan equation**

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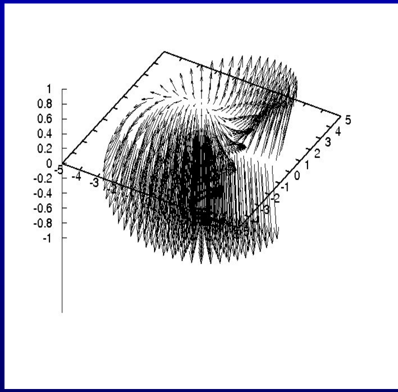
III. Field Models with Topological Charge
o - Recent Motivation
- Recent two experiments observed magnetic pattern in helimagnet MnSI on a plane of *topological* origin (Lee et al PRL 2009, Neubauer et al PRL 2009]
- Expected to be explained by topological Skyrmion (for magnetic moment) on the map: $S^2 \longrightarrow S^2$
Induced by competing symmetric (Heisenberg) and asymmetric (Dzyaloshinsky-Moriya)(DM)

spin interactions, due to *non-inversion-Symmetric* nature of MnSi.

• **Our Result :** (A Kundu Phys. Lett B 1982; A Kundu arXiv: 0906.3188 (2009))

We discovered earlier, using a fascinating simple geometry: *Exact Non-circular Symmetric* Skyrmion saturating Bogomolny bound with $Q = N = 1, 2, \ldots$ (*No other such exact soln. in 2D with noncircular symmetry is known!*: Belavin-Polyakov's 2D Skyrmion is circular symmetric.)



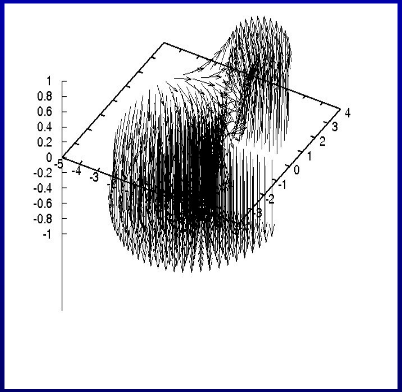


Figure 5: Skyrmion spin texture described by the exact solution $\mathbf{n}^*$, with topological charge: a) $Q = -1$ and b) $Q = -3$

Current/Future Plan:

• We believe that, due to *non-Inversion Symmetry* in MnSI, *non-Circular Symmetric* Skyrmion will be most suitable to explain the observed recent magnetic structure in MnSI

• We hope that adding DM term to our earlier exact model, we will be able to get *noncircular Skyrmions* with $Q = N, \ N = 1, 2, \ldots$, which can guide the future experiment on MnSI: to look for the *non-Circular symmetry* as well as *higher topological charge* in magnetic moment configuration.

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Thank You !!!

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