# Integrable two-fold hierarchy of perturbed AKNS and KN systems through nonlinearization of linear equations

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**Plan of Talk**:

• Nonlinearizing linear equations to Integrable Systems of both AKNS-ZS and KN type
• Perturbations (with nonholonomic constraints) preserving integrability
• Richer structure: - two-fold Integrable Hierarchy,
- Accelerating exact Solitons
• Possible application/ extension

$II$. **Nonlinear Integrable systems**

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$II.$ **Nonlinear Integrable systems**
1)**Soliton solutions**
2)**Lax pair** : Flatness condition:

$$\boxed{\mathbf{U}(\lambda)_{\mathbf{t}} - \mathbf{V}_{\mathbf{n}}(\lambda)_{\mathbf{x}} + [\mathbf{U}(\lambda), \mathbf{V}_{\mathbf{n}}(\lambda)] = \mathbf{0}}$$

giving
3) **Hierarchy of Integrable Nonlinear eqns.** for $n = 1, 2, \ldots$:

Integrable NonLin. eqn = Linear eqn. + Integrable Nonlinearityi

Examples:
i. KdV: $u_t - u_{xxx} + 6uu_x = 0$
ii. MKdV: $u_t - u_{xxx} + 6u^2u_x = 0$
iii. NLS: $iq_t - q_{xx} + 2|q|^2q = 0$ , comp. conj.
iv. Derivative NLS: $iq_t - q_{xx} + 2(|q|^2q)_x = 0$, comp. conj.
v. Sine-Gordon: $\theta_{xt} - \sin\theta = 0$
etc.

• **Our Question**:
1) Knowing only Linear eqn. Can we find the Integrable Nonlinearity for above systems ?
2) Can we perturb these int. systems preserving integrability to get new int. systems?

• **Our Motivation**:
1) - The idea already exits (Sato, Zakharov, Calogero-Degasperis-Manakov)
- *Soliton* = balance: *Lin dispersive* term + *NonLin term*
Hence *Lin term* must keep hidden *Info about NonLin term* !

2) - Idea exists (Melnikov'1988), but complicated for application
-Together with well known
*i. Camassa-Holms*
*ii. Degasperis-Pocesi*
Some strange new integrable eqns discovered:
*iii. KdV6 (2007)*
*iv. Lenells-Fokas DNLS (2009)*
v. mKdV-SG (2009)
vi. Higher mkdv (2002)
They must be particular cases of general integrable perturbations of AKNS-ZS and KN systems

**Our Nonlinearization Scheme** will be based on simple physical concept of **scaling Dimension**
*Note*:
1) Lax eqns.

$$\Phi_x = U\Phi,\ \Phi_t = V_n\Phi$$

gives scaling dimensions (denoted by $[\ \cdot\ ]$) of Lax pair $(U, V_n)$:

$$\boxed{[\mathbf{U}] = [\mathbf{L^{-1}}] \equiv \mathbf{1}, \quad [\mathbf{V_n}] = [\mathbf{T^{-1}}]}$$

1) *Linear eqns. we start from*

$$iq_t - q_{xx} = 0, \quad iq_t^* + q_{xx}^* = 0,$$

$$u_t - u_{xxx} = 0,$$
$$\theta_{xt} = 0$$

fixes scaling dimension

$$\boxed{[\mathbf{T^{-1}}] = [\mathbf{L^{-2}}] \equiv \mathbf{2}, = [\mathbf{L^{-3}}] \equiv \mathbf{3}, = [\mathbf{L^{-1}}] \equiv \mathbf{1}, \ldots \to [\mathbf{V_n}(\lambda)] = [\mathbf{T^{-1}}] = [\mathbf{L^{-n}}] \equiv \mathbf{n}}$$

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2) Suppose:

$$U(\lambda) = U_L(\lambda) + U_{NL}(\lambda), \; V_n(\lambda) = V_L(\lambda) + V_{NL}(\lambda)$$

Naive ”linear Lax pair”: $U_L, V_L,$

$$(U_L)_t - (V_L)_x = 0 \longrightarrow \quad \text{Above Lin. eqns}$$

(without [,] term )

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*Example*: For Lin. Schrödinger eqn. (LSE)+ complex conjugate:

$$\mathbf{U_L}(\lambda) = \mathbf{i}(\lambda\sigma_3 + \mathbf{U}^{(0)}), \quad \mathbf{V_L} = \sigma^3\mathbf{U}_{\mathbf{x}}^{(0)}, \quad \text{where} \quad \mathbf{U}^{(0)} = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix},$$

Check ! LSE (for q and $q^*$):

$$iU_t^{(0)} - \sigma^3 U_{xx}^{(0)} = 0$$

$\lambda$ - const. parameter, Paui matrix $\sigma^3$ to get $trU = 0.\ trV = 0$ ,
3) Identify *Building Blocks of Lax operators* (BB) as
$BB = \{\lambda, U^{(0)}, \}$ and x-derivatives like $(U^{(0)})_x$ .

4) Construct: *NonLin Lax pair part*

Recall: $[U(\lambda)] = [U_L(\lambda)] = 1$ *Hence* by observation

$[\lambda] = [U^{(0)}] = [q] = [q^*] = 1$, $[U_x^{(0)}] = 2$

i) Space-Lax operator ($[U] = 1$ operator):

$$\mathbf{U}(\lambda) \equiv \mathbf{U_L} = \mathbf{i}(\lambda\sigma^{\mathbf{3}} + \mathbf{U}^{(\mathbf{0})})$$

Since there is no other $[\,\cdot\,] = 1$ operators from BB !

ii) Time-Lax operator ( [V]=2, operator) from BB $\lambda, U^{(0)}$ (with $[\,\cdot\,] = 1$ objects ):

by their *product - powers - derivative* :

$$\mathbf{V_2}(\lambda) = \mathbf{V_L} + \underbrace{(2i\lambda^2 - (U^{(0)})^2)\sigma^3 + 2i\lambda U^{(0)}}_{\mathbf{V_{NL}}}$$

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NB: No other $[\,\cdot\,] = 2$ operators!

ii). Coefficients are fixed uniquely by Flatness Condition

*Hence we get exactly NLS Lax pair ! by simple dimensional analysis!*

*Nonlinear terms* $2(U^{(0)})^3 \rightarrow 2|q|^2 q$ added to starting LSE, i.e get the int.

# NLSE!

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**Note**:

1) This Scheme is prototype for all AKNS-ZS family with same $U(\lambda)$:
Examples: i) Constructing similarly $[V_n] = n$ from same BB can derive NLS-hierarchy and from $[V_3] = 3$ get KdV and mKdV eqns.

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•**Derivative NLS**:

This scheme is extendible also to KN family:
i) Suppose new scaling dim. : $[q] = [\lambda] = \frac{1}{2}$
ii) Construct Lax pair: $[U(\lambda)] = 1, [V_2(\lambda)] = 2$ out of same BB $\{\lambda, U^{(0)},$ their x-derivatives $\}$, but with new $[\,\cdot\,] = \frac{1}{2}$,
iii) This constructs

$$U = i(\lambda U^{(0)} + \lambda^2 \sigma^3),$$

Similarly for

$$V_2(\lambda) = \lambda(\sigma^3 U_x^{(0)} + i(U^{(0)})^3) - i\sigma^3 (U^{(0)})^2 \lambda^2 + 2iU^{(0)} \lambda^3 + 2i\sigma^3 \lambda^4$$

This is nothing but DNLS Lax-pair giving DNLS eqn:= $iq_t - q_{xx} + 2i(|q|^2 q)_x = 0$

from the same LSE= : $iq_t - q_{xx} = 0$ !

from the same LSE= : $iq_t - q_{xx} = 0$ !
6) This is a Simple Algorithmic method for constructing $U, V_n$ pair, based on basic physical concept of dimensionality

ii) **Integrable perturbation through Nonholonomic deformation**

Notice:

Actually one can add $[G_{(1)}] = 3$ operator in $[V_2(\lambda)] = 2$ ! like $\lambda^{-1} G_{(1)}$, where $G_{(1)}$ contains *perturbing* fields with higher scaling dimension (i.e. we go beyond BB construction)

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*known Integrable eqn = g(x,t) (perturbing function)* with *Non-Holonomic constraint* on $g$: ( involving only $\partial_x \sim$ "momentum")$\hat{L}(g, g_x, g_{xx}; q) = 0$ (Not evolution eqn.)

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*Example*: Perturbed (forced) NLS:

$$\mathbf{iq_t - q_{xx} - 2|q|^2q = g,}$$

with *Nonholonomic constraint*:

$$\mathbf{g_{xx}q - g_xq_x - 2q^2(qg^* - q^*g) = 0.}$$

Physical application:
This constraint can be split up as set of eqns.

$$\mathbf{g_x = -2iNq, \ N_x = i(qg^* - q^*g)}$$

Idea used in communication by optical soliton through boosting by doped media (Laser) ($q = E$- Electric field, $g$-polarization, and $N$- population inversion of media=atom )

**Accelerating soliton**

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**Accelerating soliton**
Solitons derived easily from NLS but with deformation of soliton velocity, frequency etc., (can make them time-dependent !)
Therefore can get
*Accelerating NLS Soliton*.
1. Soliton velocity $V = v_0 + v_{def}(t)$ 2. Wave frequency $\Omega = \omega_0 + \omega_{def}(t)$ where $\omega_{def}(t), \ v_{def}(t)$ are fixed by perturbing functions at space-boundaries: $g(x,t)|_{|x| \to \infty}$.

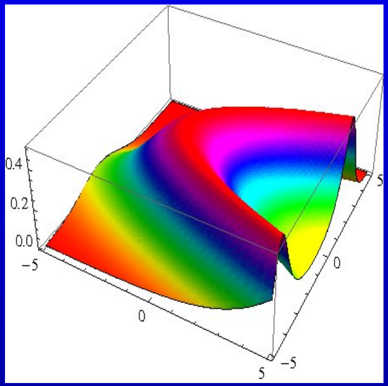
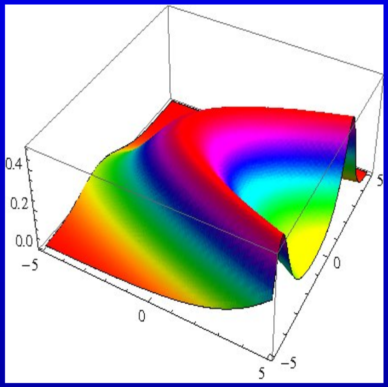



Figure 1: Exact accelerating soliton for int-perturbed NLS eqn. $|q|$

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**Two-fold integrable hierarchy** :
1) 1st hierarchy:
Standard *higher order NLS* eqns. with *same perturbation* (as above)
2) 2nd hierarchy
Same NLS with hierarchy of constraints with coupled *deforming* operators with higher scaling dimensions $[G_{(n)}] = (2+n)$

$$\mathbf{G_x^{(n-1)} = i[U^{(0)}, G^{(n-1)}] + i[\sigma_3, G^{(n)}], \quad G_x^{(N)} = i[U^{(0)}, G^{(N)}]}$$

$n \in [2, N]$
NB: For NLS :
Such deformed hierarchy can be used for
1. multiple boosting of optical solitons in multiply-doped media:

**Integrable Nonholonomic Deformation of other AKNS-ZS systems**

ii) **Int-perturbed KdV**:

$$\mathbf{u_t - u_{xxx} - 6uu_x = g,}$$

with Non-Holonomic constraint $g = w_x$:

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$$\mathbf{u_t - u_{xxx} - 6uu_x = g,}$$

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$$\mathbf{w_{xxx} + 4uw_x + 2u_xw = 0}$$

Note: 1). Equivalent to 6th- KdV eqn. discovered recently + its 2-fold hierarchy !
2). Other particular cases $\longrightarrow$
**Camassa-Holm** & **Degasperis-Procesi** eqn.

Note: 1). Equivalent to 6th- KdV eqn. discovered recently + its 2-fold hierarchy !
2). Other particular cases $\rightarrow$
**Camassa-Holm** & **Degasperis-Procesi** eqn.
Accelerating, soliton ! (Idea pioneered in *Bumeron Solution* (Degasperis+Calogero (1977)))

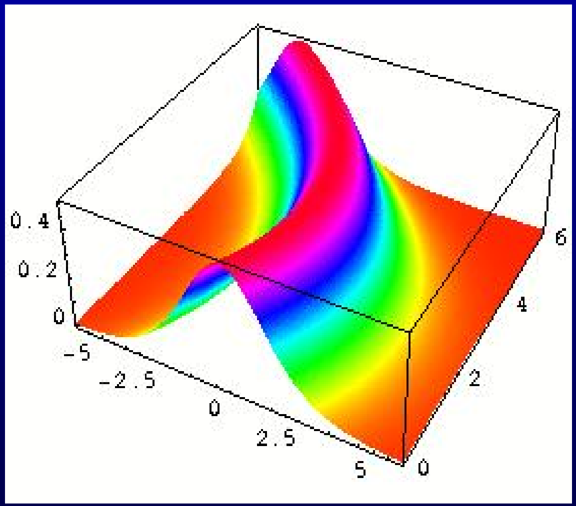


Figure 2: Exact accelerating soliton for Int-perturbed KdV $u(x,t)$. ( Bending of soliton in the (x,t)-plain shows acceleration )

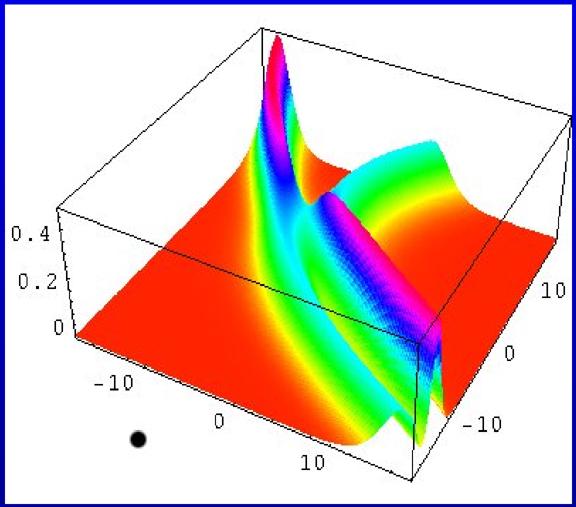


Figure 3: Exact 2-soliton $u(x,t)$ for perturbed KdV. ( Usual elastic scattering, but with unusual dynamics of soliton acceleration).

ii) **Int-Perturbed mKdV**:

$$\mathbf{v_t - v_{xxx} - 6v^2 v_x = w_x(t,x), \quad w_{xx} - 2v(c^2(t) - w_x^2)^{\frac{1}{2}} = 0.}$$

with two-fold Int-hierarchy!

Solitons similar to pertured NLS soliton:

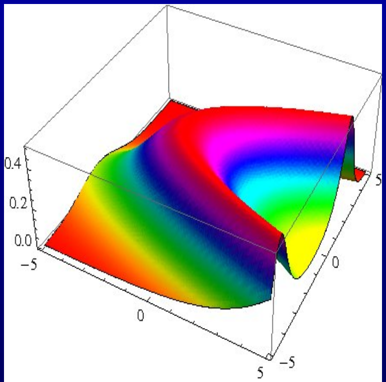


Figure 4: Exact accelerating soliton for perturbed mKdV eqn. $v(x,t)$

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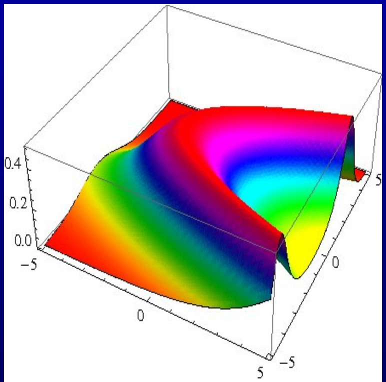


Figure 4: Exact accelerating soliton for perturbed mKdV eqn. $v(x,t)$

iii) **Int-Perturbed sine-Gordon** (in light-cone coordinates):

$$\theta_{xt} = \mathbf{e}(\mathbf{t})(\alpha \sin \mathbf{2}\theta + \beta \cos \mathbf{2}\theta), \quad \alpha_{x} = \sin \mathbf{2}\theta, \ \beta_{x} = \cos \mathbf{2}\theta,$$

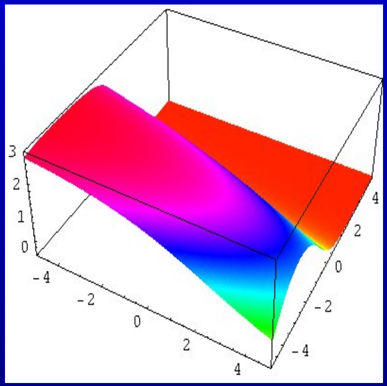


Figure 5: Exact accelerating kink for perturbed SG eqn. $\theta(x - v(t))$

Finally we derive int. perturbation for DNLS (extending our scheme to the KN systems) as

**Integrable Deformed DNLS**:

$$\mathbf{i}\frac{\mathbf{1}}{\mathbf{2}}\mathbf{q}_{t} - \mathbf{q}_{xx} + \mathbf{2i}\epsilon(|\mathbf{q}|^{\mathbf{2}}\mathbf{q})_{x} + (-\mathbf{2w} + \epsilon\mathbf{gq}) = \mathbf{0},$$

## Perturbing functions $w(x,t), g(x,t)$ with NonHol. constraints

$$\mathbf{g_x(x,t) = i\epsilon(qw^* - q^*w),\ w_x(x,t) = 2ia(t)q.}$$

This integrable DNLS eqn allows

i) Exact N-soliton with acceleration !

ii). Application to NL optics (Boosting of optical soliton)

iii) Two-fold Int-hierarchy

- Remarkably,

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• Remarkably, we find recently proposed **Lenells-Fokas** (LF) equation , as lowest member of this hierarchy (putting $q = u_x$ ):

$$\mathbf{i(\frac{1}{2}u_{xt} + au) - \alpha u_{xx} + cu_x + \epsilon a|u|^2 u_x = 0}$$

NB: Here for time-dependent coeff. $a(t), c(t)$ one gets *accelerating soliton* in the LF eqn.

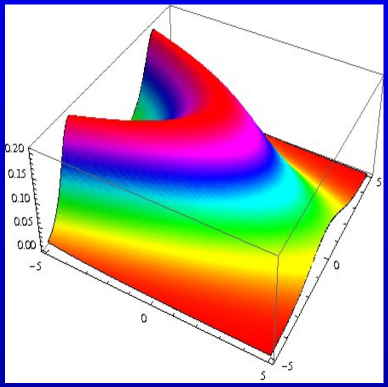


Figure 6: Exact accelerating soliton $|q(x,t)|$ in perturbed DNLS and LF eqn.

*Similarly*: 6) Integrable Perturbed 3-Wave Eqn.

$$u_{1t} + v_1 u_{1x} - i\epsilon \bar{u}_2 u_3 = p_1 a_{21}$$

and cyclically for $1 \to 2 \to 3$.
Here $u_j,\ j = 1,2,3$ and conjugate $\bar{u}_j$ are 3-wave fields; $v_j, ajk$ const. coefficients and

$p_j, p_j^d \; j = 1, 2, 3$ are *perturbing* fields with constraint

$$p_{1x} - i(u_3\bar{p}_2 + \bar{u}_2 p_3)A_{123} - iu_1(p_1^d - p_1^d),$$

$$p_{1x}^d + (u_1\bar{p}_1 + \bar{u}_1 p_1)a_{12} + (u_3\bar{p}_3 + \bar{u}_3 p_3)a_{13}$$

**Failed attempts** to construct integrable perturbations:

$p_j, p_j^d \; j = 1, 2, 3$ are *perturbing* fields with constraint

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**Failed attempts** to construct integrable perturbations:

i). (2+1)-Devey-Stewartson, KP

ii) (0+1)- Painlevé Transcendents (PT) (PI-PVI): y(t)

PI:

$$y^{''} = 6y^2 + t$$

PII:

$$y^{''} = 2y^3 + ty + const.$$

PIII-PVI (more complicated to reproduce here)

**Success**:

Finally we succeeded for PII:

$$\boxed{\mathbf{y^{''} = 2y^3 + ty + p(t) + const, \quad p^{''} + 2((p^{'})^2 + c)^{\frac{1}{2}} y = 0}}$$

which can be extended to a deformed Hierarchy!

**Appeal**:
Anyone having ideas to extend this scheme to (0+1)- (PI-PIV) or to (2+1)-models are kindly requested to contact me!

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# Thank You !

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