Generating integrable systems including new hierarchy by Nonlinearization

Anjan Kundu
Theory Group & CAMCS, Saha Institute of Nuclear Physics
Calcutta, INDIA
anjan.kundusaha.ac.in

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Integrable systems exclusive class of *nonlinear equations* with rich structures:

- *Exact n-soliton,*
- *Hierarchy of integrable equations*
- *Infinite number of* $C_n, \ n = 1, 2, \ldots$
- Nonlinearity *diverse* but of *specific form*:

1. NLS: $iq_t = q_{xx} + 2|q|^2 q$
2. derivative NLS (DNLS): $iq_t + q_{xx} + 2(|q|^2 q)_x = 0$
3. KdV: $q_t = q_{xxx} + 6qq_x$
4. modified KdV (mKdV): $q_t = q_{xxx} + 6q^2 q_x$ etc.

[NonLin Eq= Lin Eq + *integrable nonlinearities*]

• Remains *mystery*

how to find a priory such *integrable nonlinearities*,
i.e. *nonlinearize a linear equation* to make it integrable!!

• The question is avoided in almost all studies !! (recent exception: Pinotsis, JNlMath Phys.2007)

Most Studies start from *given nonlinear equation*
-Explore properties (symmetries, solutions, geometry,P-test, r-matrix, YBE)

- build up structures (Lax pair, prolongation, consv. quant., Hamil., bi-Hamil.)

•Our aim

1. Start with Lin Eq. and *nonlinearize* it to

i) Known integrable hierarchies of AKNS and KN family

ii). Find new *physically interesting* integrable systems (Everybody's dream !)

Start from Lin Eq like $iq_t = q_{xx}, \quad q_t = q_{xxx}$ etc
(and its conjugate for $r(t,x)$),

Note:
Nonlinearization through Scaling Dimension (SD):
Example: $iq_t = q_{xx}$ – SD of each term $[q][L]^{-2}$, can allow
(assuming $[q] = [r] = [L]^{-1}$ and a dimension-less coupling)
additional nonLin terms like(Check!)

$$q^2r, qr^2, qr_x, qq_x, q^3, r^3$$

Only first possibility gives $q|q|^2$ for $r = q^*$: $-- >$ integrable NLS .
Conclusion:

- Nonlinearizing Lin Eq directly *not unique*
-One needs more refined method for filtering out *integrable cases*
Therefore
•**Our strategy** - Stick from start to structures akin to integrable systems:
Define *linear Lax pair* $U^{(l)}, V^{(l)}$ for Lin system and *nonlinearize* them to genuine Lax pair
$U, V \in sl(2)$ with Lax equations: $\Phi_x = U\Phi,\ \Phi_t = V\Phi$.
nonLin Int Eq obtained from Compatibility or *flatness condition*: $U_t - V_x + [U, V] = 0$.

Start from Lin Eq

$$U_t^{(0)} = U_x^{(0)}, iU_t^{(0)} = \sigma_3 U_{xx}^{(0)}, U_t^{(0)} = U_{xxx}^{(0)},$$

etc. Combining q, r in a matrix

$$U^{(0)} = q\sigma^+ + r\sigma^- = i \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}$$

1. By asy observation, cast Lin Eq as *linear* flatness condition $U_t^{(l)} - V_x^{(l)} = 0$, by defining naive Lax pair

$$U^{(l)} = i\lambda\sigma_3 + iU^{(0)},$$

and

$$V_2^{(l)} = \sigma_3 U_x^{(0)},\ V_3^{(l)} = iU_{xx}^{(0)}$$

etc.

- Parameter λ, can be put to zero here without any loss,
(Its significance will be clear in the process).
Note:
i) All lin Lax pair constructed from
$\lambda, U^{(0)}$, and its derivatives only (our building block) .(Here they enter only linearly)!!
Nonlinearization criteria
ii) Nonlinear Lax pair should also be build from the same building block (though through NonLin combination guided by the SD

Dimensional argument:

For any Lax pair U, V, dimensional unit (check from Lax equations):
$[U] = [L]^{-1}, \ [V] = [T]^{-1}$.
Assume same for linearized; $U^{(l)}, V^{(l)}$,

Therefore from above form of $U^{(l)}$: Length or the Scaling Dimension (SD): $[\lambda] = [U^{(0)}] = 1$,
Hence for fields $[q] = [r] = 1$,
(with notation $n := [L]^{-n}$).

Note:

- Since Lin Eq puts no restriction on SD of fields, this input is not unique
- We will show : different SD for fields leads to another Lax pair.
- Lin Eq however link $[T] = [L]^N$: hence $[V_N] = N$, (since $[\frac{\partial}{\partial x}] = 1$).

(Gauge equivalent systems) Fundamental Int Eq $\rightarrow$ GE Eq (Examples: i) NLS $-->$ KE Eq , ii) DNLS $-->$ Chen-Lie-Liu Eq etc [K- JMP'84]

Therefore we focus on NLS, KdV DNLS etc. belonging to AKNS or KN spectral problem only.
(gauge equiv. systems can be obtained by simple transformations).

i) We start with Lin Schröd Eq (LS)$iq_t = q_{xx}$.

Puzzle! :

two well known Int Eqs: NLS and DNLS have same Lin LS!

•: *How to generate different Int systems from the same Lin Eq ?*

Simple answer

Attach different SD for their fields $--\, >$ two different forms of Lax pairs: one $\in$ AKNS and other $\in$ KN spectral problems)

I. First consider fields with $[q] = [r] = 1$.

Construction of space-Lax operator $U(\lambda)$

Since $U^{(l)}$ to be built from $U^{(0)}$ and λ, all of them having SD 1, $U^{(l)}$ can not be deformed any more. Hence,

$U(\lambda) = U^{(l)} = i\lambda\sigma_3 + iU^{(0)}$,.

(true for all higher Eq)= well known AKNS Lax operator)

Construction of time-Lax operator $V_2(\lambda)$

Since SD $[V_2] = 2$ for 2nd order Eq, to Lin $V_2^{(l)}$ (known) additional nonLin terms $V_2^{(nl)}$ following our Construction criteria must be built from $U^{(0)}$, λ through products and powers with combined

SD $2 = [\lambda^2] = [\lambda][U^{(0)}] = [(U^{(0)})^2]$.
Therefore (considering $V_2(\lambda) \in sl(2)$) unambiguously
$V_2^{(nl)} = 2i\lambda^2\sigma_3 + 2i\lambda U^{(0)} - i\sigma_3(U^{(0)})^2$,
generating finally
$V_2(\lambda) = V_2^{(l)} + V_2^{(nl)}$.
Note: -Fixing the dim-less numerical factors of different terms in $V_2(\lambda)$ goes beyond scope of dim argument.
However is achieved from the flatness condition of $U(\lambda), V_2(\lambda)$)

NonLin Eq
Flatness condition $-->$ nonLin Eq =LS + nonLin term $2\sigma_3(U^{(0)})^3$

- $+\ 2|q|^2q$ for $r = \pm q^*$, i.e. NLS !!.

This completes our nonlinearization procedure !
ii). **NonLin-earization of 3rd order Eq** $[q] = 1$ -Start with Lin Eq
$q_t = q_{xxx}$
-Follows similar procedure with same dim argument,
Same x-Lax operator $U(\lambda)$ constructed above (AKNS)

Construction of t-Lax operator $V_3(\lambda)$

Since SD $[V_3] = 3$ for 3nd order Eq, to Lin $V_3^{(l)}$ (known) additional nonLin terms $V_3^{(nl)}$ must be constructed out of building blocks $(\lambda, U^{(0)}$ and its derivatives), through possible nonLin combinations with SD $3 = [\lambda^3] = [\lambda][U_x^{(0)}] = [\lambda][(U^{(0)})^2] = [\lambda^2][U^{(0)}] = [U^{(0)}][U_x^{(0)}] = [(U^{(0)})^3]$. This with $V_3(\lambda) \in sl(2)$ gives nonLin part unambiguously, generating finally

$V_3(\lambda) = V_3^{(l)} + V_3^{(nl)} =$ 3rd AKNS

NonLin Eq

Flatness condition fixes i) numerical factors ii)yields

nonLin Eq = 3rd ord. Lin Eq $(U_t^{(0)} = U_{xxx}^{(0)})$

+ nonLin term $((U^{(0)})^2 U_x^{(0)} + U_x^{(0)}(U^{(0)})^2)$

$= q_t = q_{xxx} + 6(qr)q_x$

$-- >$ KdV (for $r = 1$) & mKdV (for $r = \pm q$)

N-th order Eq :

$U(\lambda)$ same.
for both are already $V_N(\lambda)$ with SD N to be constructed (remember our buildind blocks !) by nonLin combinations like $(\lambda)^k(\frac{\partial}{\partial x})^l(U(0))^m$.
with SD constraint: $k+l+m=N$. Partitioning fixes total number of such terms as $T_N = 1+\frac{1}{2}N(N+1)$,(+1 for Lin part !)
(check : $T_2 = 4, \ T_3 = 7$

II. **Int KN hierarchy**
-We start from same set of Lin Eq,
- But Nonlinearize them assuming different SD: $[\lambda] = [U^{(0)}] = \frac{1}{2}$, i.e. $[q] = [r] = \frac{1}{2}$.

Construction of x-Lax operator $U(\lambda)$
Can only be $U(\lambda) = \lambda U^{(l)}$ (since SD $1 = \frac{1}{2} + \frac{1}{2}$
(KN Lax opr.)
ii) *Construction of t-Lax operator* $V_2(\lambda)$
Same LS: : $[V_2(\lambda)] = 2$ to be formed out of
$(\lambda, U^{(0)}$ (SD=$\frac{1}{2}$) and derivative (SD=1))

as match $2 = [\lambda][U_x^{(0)}] = [\lambda][(U^{(0)})^3] = [\lambda^2][(U^{(0)})^2] = [\lambda^4]$.
(Note: though $[(U^{(0)})^4]$ matches, such terms without λ-dependence vanish due to compatibility).

This argument fixes time-Lax operator uniquely as KN:

$$V_2(\lambda) = (\sigma_3 U_x^{(0)} + i(U^{(0)})^3)\lambda - i\sigma_3(U^{(0)})^2\lambda^2 + 2iU^{(0)}\lambda^3 + 2i\sigma_3\lambda^4$$

NonLin Eq

Flatness condition $-->$ nonLin Eq = LS + nonLin term $i((U^{(0)})^3)_x$=DNLS (for $r = q^*$, giving $(2|q|^2 q)_x$).

Therefore *same LS yields thus DNLS equation* Resolving the puzzle!

ii) *Construction of higher order Eq* Same lin Eq we used gives now Different set of int. Eq (due to different SD!

Construct: $V_N(\lambda)$ with SD N same as above but With SD constraint $\frac{1}{2}(k+m) + l = N$, w This gives total no . of terms $\tilde{T}_N = 1 + N^2$ terms.
(check : $\tilde{T}_2 = 5(DNLS)$, $T_3 = 10$)

New Int hierarchy

We discover *new Int extensions deforming the AKNS hierarchy*

The Idea

-Extend nonlinearization for time-Lax operator $V(\lambda)$ to include -ve powers of λ, .

1. Example: deform $V_2(\lambda)$ additional term $V_2^{(nl-ext)} = \frac{i}{2}\lambda^{-1}G$ with G matrix function of SD 3. We may construct it explicitly as before However flatness condition!
yields *New Int NLS with a source* G, having a differential constraint:

$$iq_t - q_{xx} - 2(qr)q = g(t,x), \quad G_x = i[U^{(0)}, G]$$

where $g(t,x) = G_{12}$,

2. Note: Novelty: Such source can be deformed further (for the same NLS with force:
by adding in $V_2^{(nl-ext)}$ $\frac{i}{2}\lambda^{-2}F$, with matrix function F of SD 4.

-This leads to another new Int extension:
replace constraint with new one

$$G_x = i[U^{(0)}, G] + i[\sigma_3, F], \ F_x = i[U^{(0)}, F].$$

We conclude: This process of Int deformations by adding higher -ve powers of λ possible for any V_N, while keeping its space-Lax operator

the same.
Hence each N produces its own int hierarchy of forced eqns !

• We demonstrate it for V_3 with a similar addition: $V_3^{(nl-ext)}=\lambda^{-1}G_x+\lambda^{-2}\tilde{F}$,
Result: interesting Int deformation of 3rd AKNS Eq with a source

$$q_t - q_{xxx} - 6(qr)q_x = g_x(t,x)$$

with same constraint on G, F as above
1) for $r = 1, q = u -- >$ Int KdV with source g_x with constraint

$$g_{xxx} + (ug_x) + 2(ug)_x = 0$$

.
Eliminating g one can rewrite this set as 6th order KdV, [through Painlevé analysis : Karsu et al, arXiv: 0708:3247)]
New Int mKdV with source is obtained similarly at $r = q$.

Another Int hierarchy of : Source SG (SGs)

• intriguing question:
What happens when time-Lax operator solely as deformation $V(\lambda) =$

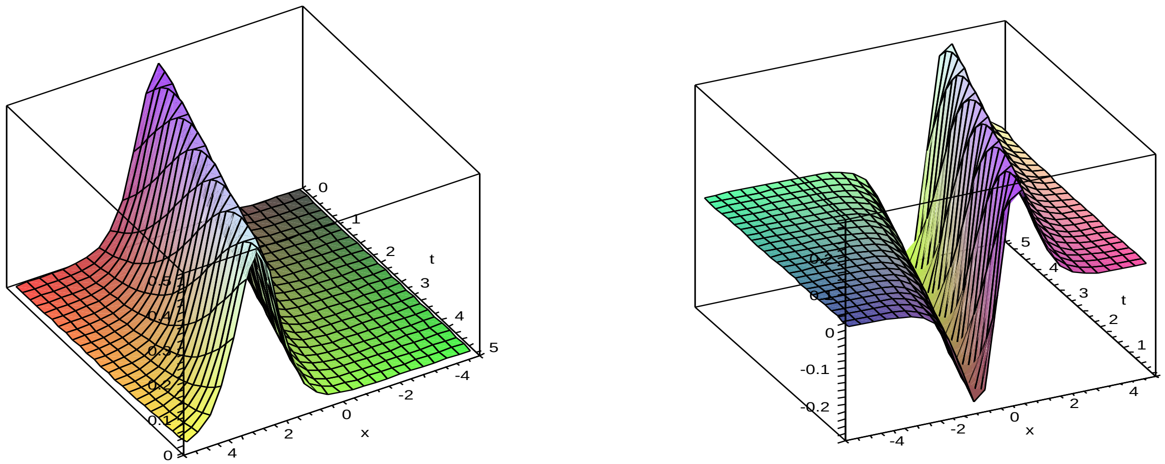

Figure 1: Exact soliton solution for the integrable KdV with source showing dynamics for a) the field $u(x,t)$ and b) the source $g_x(x,t)$. The original soliton velocity v_0 is boosted by an additional $v_1 = -\frac{c}{v_0}$ due to the source driving.

$V^{(nl-ext)}$, (dropping out both $V^{(l)}$ and $V^{(nl)}$)
Original nonlin Eq disappears!, one gets only source Eq $iq_t = G_{12}$, with constraint on the source G, F as above.
Result: i) Only one constraint G:
taking $q = r = \theta_x$ solve the constraint as $G_{12} = -i \sin 2\theta$,

i.e. surprisingly get pure SG in light-cone :
$\theta_{xt} + \sin 2\theta = 0.$
However taking two constraints one really gets a new physically interesting SGs:

$$\theta_{xxt} + \sin 2\theta = -2\phi\theta_x, \quad \phi_x = 2\theta_x\theta_{xt}$$

One can construct int hierarchy of SGs by continued deformations.!!
Remark:
1. *Thus we uniquely nonLin-rize same Lin Eq to two sets of nonLin Int Eq :* AKNS and KN Int hierarchies
2. *We discover a new Int hierarchy extending each Eq in AKNS hierarchy (by including -ve powers of* λ*)*
3. Nature of the source in Int Eq can be changed recursively by moving to the next higher Eq in the hierarchy .
4. *Exact Soliton solutions for such force eqns can be derived by standard techniques . Soliton velocity and envelope frequency get boosted under driving force.*
(depending on asymptotic of source $G_{11} \rightarrow_{x\rightarrow\infty} C$

Thank You !