Transverse Lattice QCD

A. Harindranath (SINP)

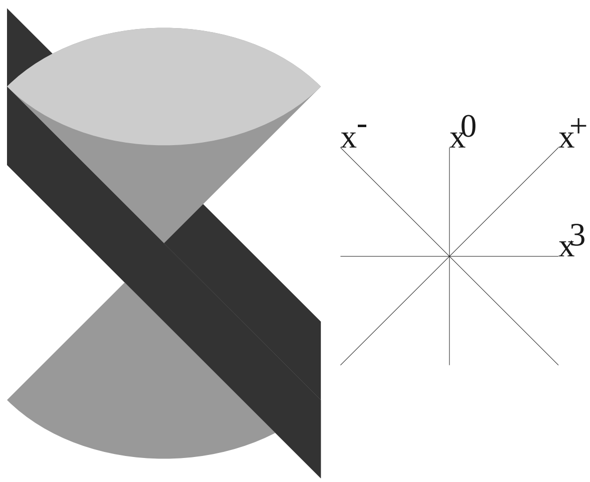
In collaboration with
Dipankar Chakrabarti, Asit De (SINP)
James P. Vary (ISU)

Plan

- Light front Dynamics: an Introduction
- Why Transverse Lattice QCD?
- Light front QCD on the Transverse Lattice
- Fermions on the Transverse Lattice
- Linearized theory
- Meson in the $q\bar{q}$ approximation
- Summary

Light Front Dynamics

Dirac, RMP (1949)



$x^{\pm} = x^0 \pm x^3,\ x^{\perp} = (x^1, x^2)$

x^+ "time", x^- longitudinal coordinate

$x^2 = x^+ x^- - (x^{\perp})^2$

$[\ x^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2\ ]$

$k.x = \frac{1}{2}k^+ x^- + \frac{1}{2}k^- x^+ - k^{\perp} \cdot x^{\perp}$

For an on mass shell particle $k^2 = m^2 \rightarrow$

longitudinal momentum $k^+ = k^0 + k^3 \geq 0$

energy $k^- = \frac{(k^{\perp})^2 + m^2}{k^+}$.

No square root

Non-relativistic structure in transverse plane

Large energy for large $k^{\perp}$, small k^+

Lorentz Symmetries

Boosts

a) Longitudinal: $x^{\pm\prime} = e^{\pm\phi} x^{\pm}$

$\rightarrow$ scale transformation

Leaves $x^+ = 0$ invariant (kinematical)

b) Transverse: Galilean boosts

Non-relativistic (kinematical)

In a relativistic theory, internal motion and the motion associated with the center of mass can be separated out at the kinematic level

Can construct boost invariant wavefunctions

Rotations

Rotation in the transverse plane kinematical

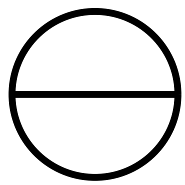
Light front helicity

Rotations about x^1, x^2 axes change $x^+ = 0$

Dynamical (like the Hamiltonian)

Apparent Triviality of the Vacuum:

Longitudinal momentum $k^+ \geq 0$



In conventional Quantum Field Theory	In Light front Field Theory
Restriction	Restriction
$\sum_i k_i = 0$	$k_i^+ = 0$

Vacuum processes receive contributions only from $k^+ = 0$. If $k^+ = 0$ is removed $(k_i^+ > \epsilon)$ Fock space vacuum $\mid 0\rangle$ is an eigenstate of the light front QCD Hamiltonian $\Longrightarrow$ Constituent picture

To build the hadron, we need not first construct the ground state (Vacuum) of the theory.

Why Transverse Lattice QCD?

- Infinitely many degrees of freedom - Need to put cutoffs
  Lattice provides a gauge invariant cutoff
- Hamiltonian provides the most direct route to wavefunctions
  Keep time direction continuous
- Theory is inherently nonlocal in x^-
  Keep x^- continuous
- Conventional ultraviolet divergences come from small $x^\perp$
  Discretize transverse space
- Retain minimal gauge invariance - gauge invariance associated with x^- independent gauge transformations
  Fix the gauge $A^+ = 0$

Many high energy experiments probe the hadron structure very close to the light cone.
Zeroth order approximation to the cut-off theory contains many nonperturbative features close to the real world, e.g. confinement.

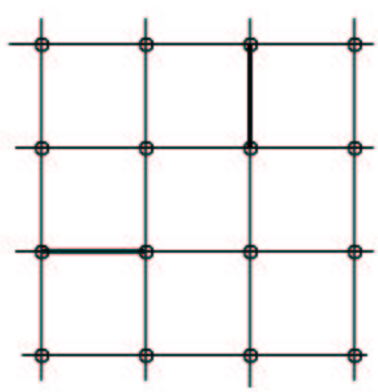
Transverse Lattice QCD

QCD Lagrangian density in the continuum

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m_f)\psi + \frac{1}{2}\mathrm{Tr}(F_{\rho\sigma}F^{\rho\sigma})$$

with $F^{\rho\sigma} = \partial^\rho A^\sigma - \partial^\sigma A^\rho + ig[A^\rho, A^\sigma], \quad A^\rho = A^{\rho a}T^a$.

Longitudinal derivative $\partial^\pm = 2\frac{\partial}{\partial x^\mp}$,
Gamma matrices $\gamma^\pm = \gamma^0 \pm \gamma^3$.
Projection operators $\Lambda^\pm = \frac{1}{4}\gamma^\mp\gamma^\pm$,
Fermion field components $\psi^\pm = \Lambda^\pm\psi$.
Choose the gauge condition $A^+ = 0$. Then A^- becomes a constrained variable.
Keeping the variable x^+, x^- continuous, discretize the transverse space, $\mathbf{x} = (x^1, x^2)$.
Replace the continuous gauge fields $A_r(x^\perp, x^-, x^+), \ r = 1, 2$ by the lattice link variable $U_r(\mathbf{x}, x^-, x^+)$ which connects $\mathbf{x}$ to $\mathbf{x} + a\hat{\mathbf{r}}$.



The constraint equations are

$$i\partial^+\psi^- = [i\alpha_r D_r + \gamma^0 m]\psi^+$$

where D_r is an appropriately defined covariant lattice derivative and

$$(\partial^+)^2 A^{-\alpha} = 2g\frac{1}{a^2}\left(J^{+\alpha}_{LINK} - J^{+\alpha}_q\right).$$

The dynamical field ψ^+ can be represented by two components such that $\psi^+(x^-, \mathbf{x}) = \begin{bmatrix} \eta(x^-, \mathbf{x}) \\ 0 \end{bmatrix}$

where η is a two component field.

The currents

$$J^{+\alpha}_q(\mathbf{x}) = 2\eta^\dagger(\mathbf{x}) T^\alpha \eta(\mathbf{x})$$

$$J^{+\alpha}_{LINK}(\mathbf{x}) = \sum_r \frac{1}{g^2} Tr\Big\{ T^\alpha [U_r(\mathbf{x}) i \overset{\leftrightarrow}{\partial^+} U^\dagger_r(\mathbf{x})$$

$$+ U^\dagger_r(\mathbf{x} - a\hat{\mathbf{r}}) i \overset{\leftrightarrow}{\partial^+} U_r(\mathbf{x} - a\hat{\mathbf{r}})] \Big\}.$$

After eliminating the constraint fields we arrive at the transverse lattice Hamiltonian $P^- = P^-_1 + P^-_2$ where P^-_1 arises from the elimination of ψ^- (hence sensitive to how fermions are put on the transverse lattice) and P^-_2 contains Wilson plaquette term and the terms arising from the elimination of A^-.

$$P_2^- = \int dx^- a^2 \sum_{\mathbf{x}} \Bigg[- \frac{1}{g^2 a^2} \sum_{r \neq s}$$

$$\Bigg\{ Trace \Big[U_r(\mathbf{x}) U_s(\mathbf{x}+a\hat{\mathbf{r}}) U_{-r}(\mathbf{x}+a\hat{\mathbf{r}}+a\hat{\mathbf{s}}) U_{-s}(\mathbf{x}+a\hat{\mathbf{s}}) - 1 \Big] \Bigg\}$$

$\uparrow$

Wilson Plaquette term

$$- \frac{g^2}{2a^2} J^{+\alpha}_{LINK} \left(\frac{1}{\partial +} \right)^2 J^{+\alpha}_{LINK}$$

$$+ \frac{g^2}{a^2} J^{+\alpha}_{LINK} \left(\frac{1}{\partial +} \right)^2 J^{+\alpha}_{q}$$

$$- \frac{g^2}{2a^2} J^{+\alpha}_{q} \left(\frac{1}{\partial +} \right)^2 J^{+\alpha}_{q} \Bigg] .$$

$\uparrow$

four $-$ fermion (current $-$ current) interaction

$\downarrow$

linear confinement in x^- direction

The presence of constraint equation for fermion field allows different methods to put fermions on the transverse lattice.

Fermions on Transverse Lattice

Fermions with forward and backward derivatives

With forward derivative for ψ^+ and backward derivative for ψ^-, in the free limit the fermionic Hamiltonian becomes

$$P^-_{fb} = P^-_0 + P^-_{hf}$$

where, the helicity nonflip term

$$P^-_0 = \int dx^- a^2 \sum_{\mathbf{x}} \Big[m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x})$$

$$-\frac{1}{a^2} \sum_r \eta^\dagger(\mathbf{x}) \sum_r \frac{1}{i\partial^+} [\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}}) \Big],$$

and the helicity flip term

$$p^-_{hf} = \int dx^- a^2 \sum_{\mathbf{x}} \frac{1}{a^2}$$

$$\eta^\dagger(\mathbf{x}) \sum_r (am\hat{\sigma}_r) \frac{1}{i\partial^+} [\eta(\mathbf{x}+a\hat{\mathbf{r}}) - 2\eta(\mathbf{x}) + \eta(\mathbf{x}-a\hat{\mathbf{r}})].$$

$\hat{\sigma}_1 = \sigma_2$ and $\hat{\sigma}_2 = -\sigma_1$. Sign of the linear mass term changes if we switch the forward and backward derivatives.

First consider the Hamiltonian without the helicity flip term. The eigenvalue equation is

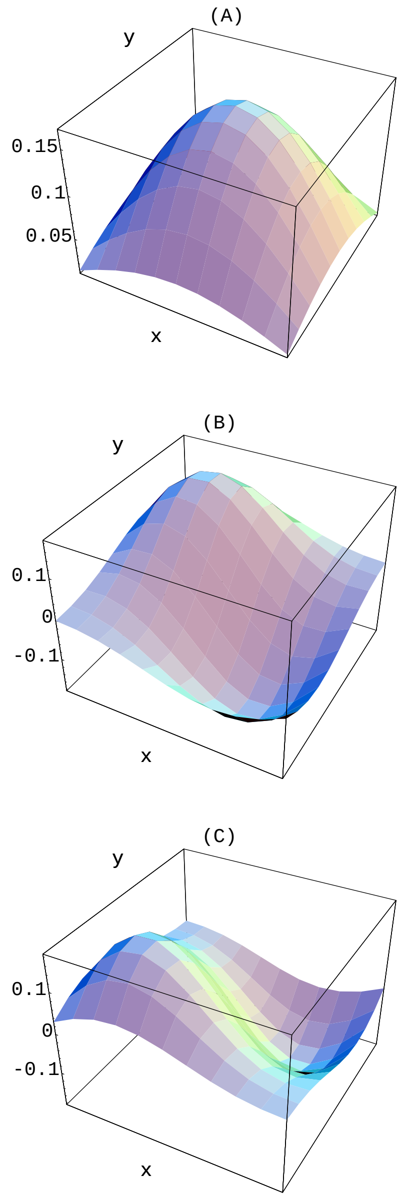
$$M^2 = m^2 + \frac{4}{a^2}\sum_r Sin^2\frac{k_r a}{2}$$

No doublers

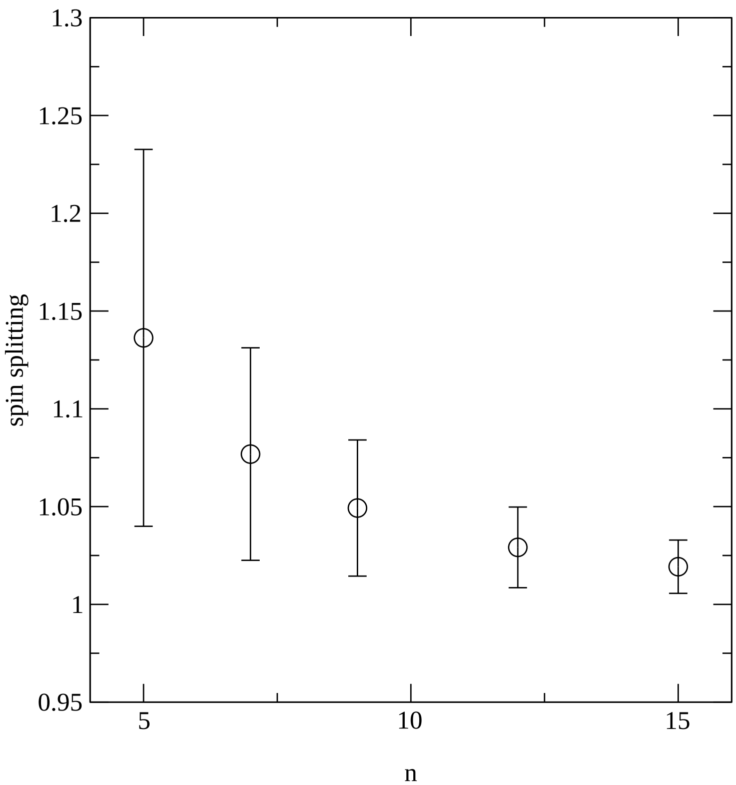
Now, consider the full Hamiltonian including the helicity flip term. The eigenvalue equation now reads

$$M^2 = m^2 + \frac{4}{a^2}\sum_r Sin^2\frac{k_r a}{2} \pm \frac{4m}{a}\sqrt{\sum_r Sin^4\frac{k_r a}{2}}$$

and is *free from fermion doubling* for physical fermions, i.e., $m < \frac{1}{a}$, the ultraviolet cutoff.



Eigenfunctions of lowest three states for the case of no doubling. n=5



Spin splitting of the ground state caused by the spin dependent interaction as a function of n.

Fermions with symmetric derivatives

With symmetric derivative for both ψ^+ and ψ^-, in the free field limit the Hamiltonian becomes

$$\begin{aligned} P^-_{sd}(\mathbf{x}) &= \int dx^- a^2 \sum_{\mathbf{x}} \Bigg\{ m^2 \eta^\dagger(\mathbf{x}) \frac{1}{i\partial^+} \eta(\mathbf{x}) \\ &+ \frac{1}{4a^2} \sum_r [\eta^\dagger(\mathbf{x}+a\hat{\mathbf{r}}) - \eta^\dagger(\mathbf{x}-a\hat{\mathbf{r}})] \frac{1}{i\partial^+} \\ &\quad [\eta(\mathbf{x}+a\hat{\mathbf{r}}) - \eta(\mathbf{x}-a\hat{\mathbf{r}})] \Bigg\}. \end{aligned}$$

- Only next to nearest neighbor interactions $\rightarrow$ odd and even lattice points decouple.
- Effective lattice spacing $2a$.
- Free particle dispersion relation for the light front energy

$$k^-_{\mathbf{k}} = \frac{1}{k^+}[m^2 + \frac{1}{a^2} \sum_r \sin^2 \, k_r a].$$

Because of the momentum bound of $\frac{\pi}{2a}$, doublers cannot arise from $ka = \pi$. However, because of the decoupling of odd and even lattices, one can get two zero transverse momentum fermions one each from the two sub-lattices. Thus, for two transverse dimensions, we can get four zero transverse momentum fermions as follows:

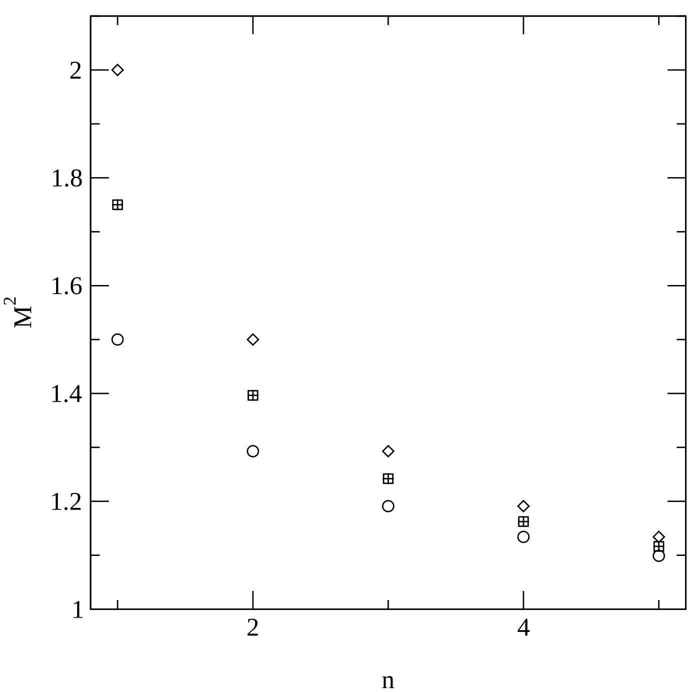
(1) even lattice points in x, even lattice points in y,

(2) even lattice points in x, odd lattice points in y,

(3) odd lattice points in x, even lattice points in y,

(4) odd lattice points in x, odd lattice points in y.

Thus we expect a four fold degeneracy of zero transverse momentum fermions.



Lowest four eigenvalues as a function of n.

Doubling and helicity flip symmetry

In lattice gauge theory in the Euclidean or equal time formalism, there has to be explicit chiral symmetry breaking in the kinetic part of the action or Hamiltonian to avoid fermion doubling. Translated to the light front formalism, this would then require helicity flip in the kinetic part since chirality *is* helicity even for a massive fermion in front form. A careful observation of all the Hamiltonians that involve fermions on the light front transverse lattice reveals that this is indeed true. In particular, we draw attention to the even-odd helicity flip transformation

$$\eta(x_1, x_2) \rightarrow (\hat{\sigma}_1)^{x_1} (\hat{\sigma}_2)^{x_2} \eta(x_1, x_2).$$

Hamiltonian invariant under this transformation shows fermion doubling and the Hamiltonian which is not invariant under this transformation is free of doublers.

Effective Hamiltonian

Because of the nonlinear constraints $U^\dagger U = 1$, $det\ U = 1$, it is highly nontrivial to perform canonical quantization of the system.
Hence Bardeen and Pearson (1976) and Bardeen, Pearson, and Rabinovicci (1980) proposed to replace the nonlinear variables U by linear variables M where M belongs to $GL(N, \mathcal{C})$, i.e., we replace $\frac{1}{g} U_r(\mathbf{x}) \to M_r(\mathbf{x})$.
Once we replace U by M, many more terms are allowed in the Lagrangian. Thus one needs to add an effective potential V_{eff} to the Lagrangian density

$$V_{eff} = -\mu^2 Tr(M^\dagger M) + \lambda_1 Tr[(M^\dagger M)^2] + \lambda_2 [det\ M + H.c] + \ldots .$$

Gauge degrees of freedom *massive* !

Residual gauge Invariance

The theory is invariant under the gauge transformations $\eta(\mathbf{x}) \rightarrow \eta'(\mathbf{x}) = G^\dagger(\mathbf{x})\eta(\mathbf{x})$ and $M_r(\mathbf{x}) \rightarrow M_r'(\mathbf{x}) = G^\dagger(\mathbf{x})M_r(\mathbf{x})G(\mathbf{x}+a\hat{\mathbf{r}})$ where $G(\mathbf{x}) = e^{-iT^a\theta^a(\mathbf{x})}$.

For infinitesimal transformation,
$G(\mathbf{x}) \approx 1 - iT^a\theta^a(\mathbf{x})$,
$\eta(\mathbf{x}) \rightarrow \eta'(\mathbf{x}) = \eta(\mathbf{x}) + iT^a\theta^a(\mathbf{x})\eta(\mathbf{x})$,
$M_r(\mathbf{x})_{pq} \rightarrow M_r'(\mathbf{x})_{pq} =$
$M_r(\mathbf{x})_{pq} + iT^a_{pl}M_r(\mathbf{x})_{lq}\theta^a(\mathbf{x}) - iM_r(\mathbf{x})_{pl}T^a_{lq}\theta^a(\mathbf{x}+a\hat{\mathbf{r}})$.

In quantum theory the gauge transformations are generated by the operator $\mathcal{G} = e^{\frac{i}{2}\sum_{\mathbf{y}} Q^a(\mathbf{y})\theta^a(\mathbf{y})}$ with

$$
\begin{aligned}
Q^a(\mathbf{y}) \quad = \quad \int dy^- \Bigg[Tr\Big\{ T^a \sum_{r'} \Big(& M_{r'}(\mathbf{y}) i \overleftrightarrow{\partial^+} M^\dagger_{r'}(\mathbf{y}) \\
& + M^\dagger_{r'}(\mathbf{y}-a\hat{\mathbf{r}}') i \overleftrightarrow{\partial^+} M_{r'}(\mathbf{y}-a\hat{\mathbf{r}}') \Big) \Big\} \\
& - 2\eta^\dagger(\mathbf{y}) T^a \eta(\mathbf{y}) \Bigg]
\end{aligned}
$$

so that $\eta(\mathbf{x}) \rightarrow \eta'(\mathbf{x}) = \mathcal{G}\eta(\mathbf{x})\mathcal{G}^\dagger$ and $M_r(\mathbf{x}) \rightarrow M_r'(\mathbf{x}) = \mathcal{G}M_r(\mathbf{x})\mathcal{G}^\dagger$.

Zeroth order Approximation for Meson

Only q and $\bar{q}$ Fock states.

Gauge invariance forces them to be on the same transverse location.

1+1 dimensional 't Hooft model: q and $\bar{q}$ interacting via instantaneous gluon exchange and self interactions →

Linear confinement in the longitudinal (x^-) direction.

Bound state equation:

$$M^2\psi_2(x) = \frac{m^2}{x(1-x)}\psi_2(x) - C_f\frac{g^2}{\pi}\int dy\,\frac{\psi_2(y)-\psi_2(x)}{(x-y)^2}$$

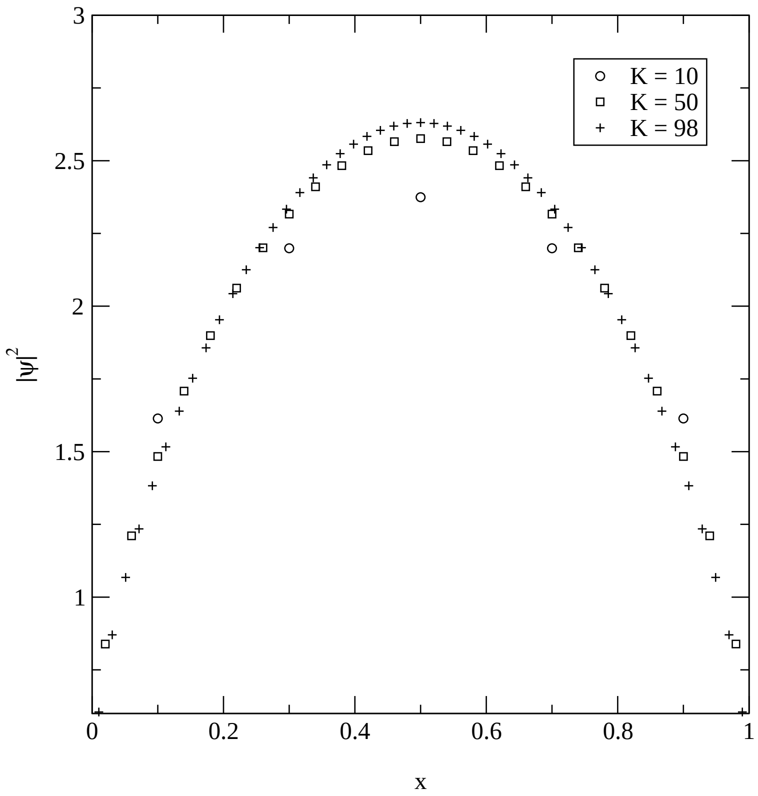
For computational purpose, use Discretized Light Cone Quantization for the longitudinal direction, $-L \leq x^- \leq +L$ and implement anti-periodic boundary condition.

Dimensionless longitudinal momentum $K = \frac{L}{2\pi}P^+$.

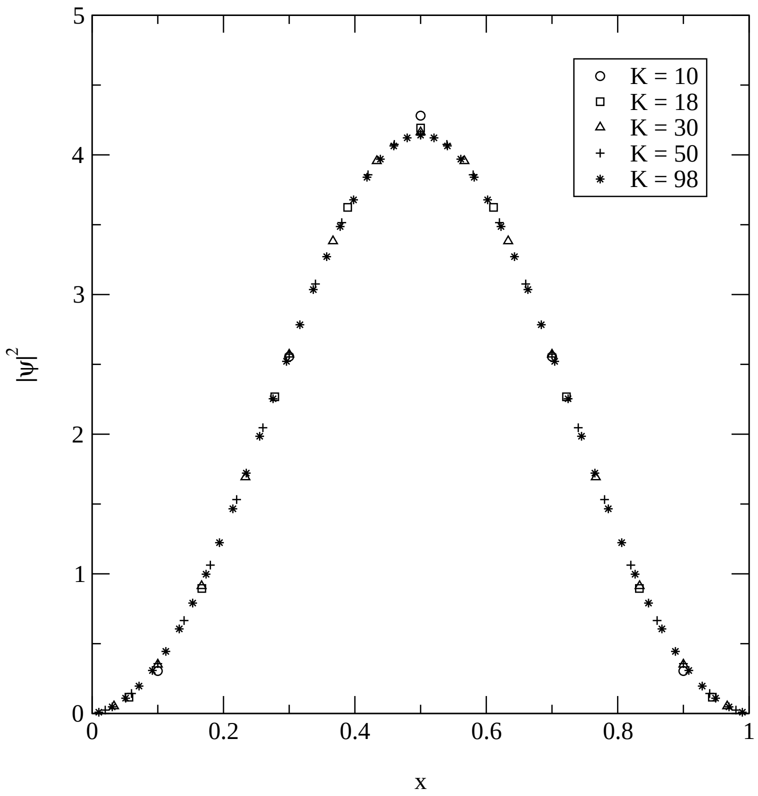
Hamiltonian diagonalization yields not only the spectra,

we also get the parton wavefunctions.

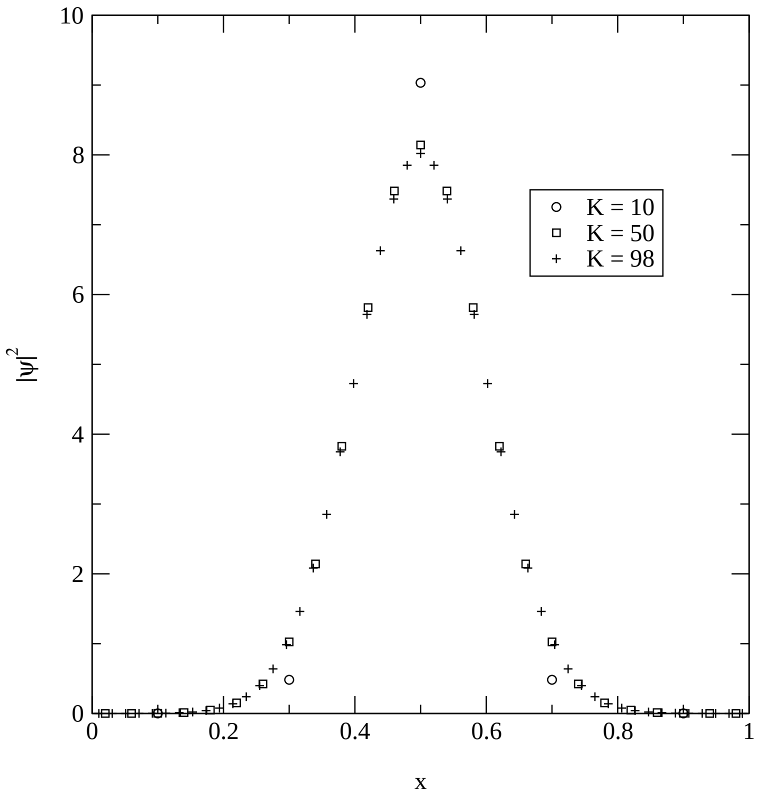
Compute structure function for weak to strong couplings.



Quark distribution function of the meson in the $q\bar{q}$ approximation. $m_f = 0.3$



Quark distribution function of the meson in the $q\bar{q}$ approximation. $m_f = 0.9$



Quark distribution function of the meson in the $q\bar{q}$ approximation. $m_f = 3.0$

Summary

After a brief introduction to salient features of light front dynamics (non-relativistic structure in the transverse plane, kinematic boosts, triviality of the vacuum, ...), motivated the study of QCD on a transverse lattice.

Presence of constraint equation on the light front allows different methods to put fermions on a transverse lattice. We mentioned two approaches: forward/backward derivatives and symmetric derivatives. Presence or absence of doublers related to a helicity-flip symmetry on the lattice.

Replacement of non-linear gauge variables by linear variables with an added effective potential results in a gauge invariant theory with massive variables.

Zeroth order approximation to meson already has many interesting features.

Dipankar Chakrabarti, Asit K. De and A.H, hep-th/0211145, to appear in Physical Review D.

Dipankar Chakrabarti, A.H and James P. Vary, work in preparation.

Work in progress ...