# An Introduction to Light-Front Dynamics for Pedestrians*

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**Abstract.** In these lectures we hope to provide an elementary introduction to selected topics in light-front dynamics. Starting from the study of free field theories of scalar boson, fermion, and massless vector boson, the canonical field commutators and propagators in the instant and front forms are compared and contrasted. Poincare algebra is described next where the explicit expressions for the Poincare generators of free scalar theory in terms of the field operators and Fock space operators are also given. Next, to illustrate the idea of Fock space description of bound states and to analyze some of the simple relativistic features of bound systems without getting into the wilderness of light-front renormalization, Quantum Electrodynamics in one space - one time dimensions is discussed along with the consideration of anomaly in this model. Lastly, light-front power counting is discussed. One of the consequences of light-front power counting in the simple setting of one space - one time dimensions is illustrated using massive Thirring model. Next, motivation for light-front power counting is discussed and power assignments for dynamical variables in three plus one dimensions are given. Simple examples of tree level Hamiltonians constructed by power counting are provided and finally the idea of reducing the number of free parameters in the theory by appealing to symmetries is illustrated using a tree level example in Yukawa theory.

## 1 Preliminaries

### 1.1 What Is a Light-Lront?

According to Dirac (1949) " ... the three-dimensional surface in space-time formed by a plane wave front advancing with the velocity of light. Such a surface will be called *front* for brevity". An example of a light-front is given by the equation $x^+ = x^0 + x^3 = 0$.

### 1.2 Light-Front Dynamics: Definition

A dynamical system is characterized by ten fundamental quantities: energy, momentum, angular momentum, and boost. In the conventional Hamiltonian form

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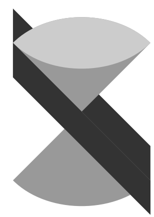

**Fig. 1.** Light-Front and Light Cone

of dynamics one works with dynamical variables referring to physical conditions at some instant of time, the simplest instant being given by $x^0 = 0$. Dirac found that other forms of relativistic dynamics are possible. For example, one may set up a dynamical theory in which the dynamical variables refer to physical conditions on a front $x^+ = 0$. The resulting dynamics is called light-front dynamics, which Dirac called *front-form* for brevity.

The variables $x^+ = x^0 + x^3$ and $x^- = x^0 - x^3$ are called light-front time and longitudinal space variables respectively. Transverse variable $x^\perp = (x^1, x^2)$. Beware that many different conventions are in use in the literature. For our conventions, notations, and some useful relations see Appendix A.

*A note on the nomenclature:*

Instead of *light-front field theory* one will also find in the literature *field theory in the infinite momentum frame*, *null plane field theory*, and *light-cone field theory*. We prefer the word light-front since the quantization surface is a light-front (tangential to the light cone).

### 1.3 Dispersion Relation

In analogy with the light-front space-time variables, we define the longitudinal momentum $k^+ = k^0 + k^3$ and light-front energy $k^- = k^0 - k^3$.

For a free massive particle $k^2 = m^2$ leads to $k^+ \geq 0$ and the dispersion relation $k^- = \frac{(k^\perp)^2 + m^2}{k^+}$.

The above dispersion relation is quite remarkable for the following reasons: (1) Even though we have a relativistic dispersion relation, there is no square root factor. (2) The dependence of the energy $k^-$ on the transverse momentum $k^\perp$ is just like in the nonrelativistic dispersion relation. (3) For $k^+$ positive (negative), $k^-$ is positive (negative). This fact has several interesting consequences. (4) The dependence of energy on $k^\perp$ and $k^+$ is *multiplicative* and large energy can result from large $k^{\perp}$ and/or small $k^{+}$. This simple observation has drastic consequences for renormalization aspects (Wilson (1990), Wilson et al. (1994)).

### 1.4 Brief History upto 1980

In the following we provide a very brief history of light-front dynamics in particle physics up to 1980 with *randomly selected* highlights. (We note that light-front has also been put to use in other areas such as optics, strings, etc.)

As we have already noted Dirac introduced light-front dynamics in 1949. In particle physics, light-front dynamics was rediscovered in the guise of field theory at infinite momentum by Fubini and Furlan (1964) in an attempt to derive " fixed $q^2$ " sum rules in the context of current algebra. Adler (1965) and Weisberger (1965) utilized infinite momentum frame in their formulation of the sum rule for axial vector coupling constant. Infinite momentum limit was also considered by Dashen and Gell-Mann (1966) for the representation of local current algebra at infinite momentum. For an introductory treatment of current algebra and light-like charges, see, Leutwyler (1969). Motivated by the work on current algebra, Weinberg (1966) studied the infinite momentum limit of old-fashioned perturbation theory diagrams and found some simplifications and also investigated the structure of bound state equations with particle truncation ("Tamm-Dancoff" approximation (Tamm (1945), Dancoff (1950))) in this limit.

In 1969, by combining the high energy ($q_0 \rightarrow i\infty$) limit with the infinite momentum limit ($P \rightarrow \infty$) Bjorken (1969) predicted the scaling of deep inelastic structure functions. Immediately following the experimental discovery of scaling in deep inelastic scattering, the celebrated parton model of Feynman came into being, which was formulated in the infinite momentum frame. Subsequently, the study of emergence of scaling in canonical field theories was carried out (see Drell, Levy, and Yan (1970)) exploiting the special features of the infinite momentum limit. Meanwhile the connection between infinite momentum limit and light-front variables became clear (Susskind (1968), Bardakci and Halpern (1968), Leutwyler (1968), Chang and Ma (1969), Jersak and Stern (1969)). This prompted the investigation of field theories in light-front quantization.

Special aspects of light-front quantization were pointed out by Leutwyler, Klauder, and Streit (1970). Kogut and Soper (1970), Bjorken, Kogut, and Soper (1971), and Neville and Rohrlich (1971) studied Quantum Electrodynamics in the light-front formulation. Cornwall and Jackiw (1971) studied the canonical equal $x^{+}$ current commutators relevant for deep inelastic scattering the phenomena of which was also studied in the context of light cone current algebra program of Fritzsch and Gell-Mann (1971). Chang, Yan and collaborators (Chang, Root, and Yan (1973), Chang and Yan (1973), Yan (1973a), Yan (1973b)) systematically investigated scalar, Yukawa, and massive vector boson theories and the connection with deep inelastic scattering.

't Hooft (1974) exploited light-front variables and light-front gauge to exhibit confinement in two-dimensional Quantum Chromodynamics (QCD) in the large

$N_c$ limit. Subsequently Marinov, Perelomov, and Terent'ev (1974) initiated the study of the spectrum of this model in the light-front Hamiltonian framework.

The intuitive picture of scaling violations in parton distributions was developed by Kogut and Susskind (1974) in the infinite momentum frame.

Investigations on the relationship between the constituent picture and the current picture in the context of classification schemes in the quark model (Close (1979)) lead to Melosh Transformation (Melosh (1974)). The nontrivial issues associated with angular momentum on the light-front came into full view with studies in light-front constituent quark models (Casher and Susskind (1973), Leutwyler (1974), Terent'ev (1976)).

The problem of $P^+ = 0$ in light-front theory (the now famous "zero mode problem") was first considered by Maskawa and Yamawaki (1976) and Nakanishi and Yamawaki (1977).

For the non-perturbative study of QCD, Bardeen and Pearson (1976) introduced the Hamiltonian transverse lattice formulation in 1976. Thorn (Thorn (1979a), Thorn (1979b), Thorn (1979c)) studied various aspects of Light-Front QCD including asymptotic freedom for the pure Yang-Mills theory.

In the late 70's and beginning of 80's Brodsky, Lepage and collaborators (Lepage and Brodsky (1980)) initiated the study of the application of light-front perturbation theory to various exclusive processes.

## 1.5 What Is Covered in these Lectures

In these lectures we hope to provide an elementary introduction to selected topics in light-front dynamics. Starting from the study of free field theories of scalar boson, fermion and massless vector boson, the canonical field commutators and propagators in the instant and front forms are compared and contrasted. Poincare algebra is described next where the explicit expressions for the Poincare generators of free scalar theory in terms of the field operators and Fock space operators are also given. Next, to illustrate the idea of Fock space description of bound states and to analyze some of the simple relativistic features of bound systems without getting into the wilderness of light-front renormalization, Quantum Electrodynamics in one space - one time dimensions is discussed along with the consideration of anomaly in this model. Lastly, light-front power counting is discussed. One of the consequences of light-front power counting in the simple setting of one space - one time dimensions is illustrated using massive Thirring model. Next, motivation for light-front power counting is discussed and power assignments for dynamical variables in three plus one dimensions are given. Simple examples of tree level Hamiltonians constructed by power counting are provided and finally the idea of reducing the number of free parameters in the theory by appealing to symmetries is illustrated using a tree level example in Yukawa theory. The notations, conventions and some useful relations are given in Appendix A. A list of review articles on light-front dynamics and a list of books where light-front has appeared are provided in Appendix B.

### 1.6 Acknowledgements

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## 2 Free Fields

In this section we consider free field theories of scalar boson, fermion and massless vector boson in the light-front formulation. In particular we discuss equal-$x^+$ commutation relations and propagators.

### 2.1 Scalar Field

The Lagrangian density expressed in light-front variables is

$$\mathcal{L} = \frac{1}{2}\partial^+\phi\partial^-\phi - \frac{1}{2}\partial^\perp\phi.\partial^\perp\phi - \frac{1}{2}\mu^2\phi^2. \tag{1}$$

The equation of motion is

$$\left[\partial^+\partial^- - (\partial^\perp)^2 + \mu^2\right]\phi = 0. \tag{2}$$

The quantized free scalar field can be written as (Leutwyler, Klauder, and Streit (1970), Rohrlich (1971), Chang, Root, and Yan (1973))

$$\phi(x) = \int_{0^+}^{\infty} \frac{dk^+ d^2k^\perp}{2k^+(2\pi)^3}\left[a(k)\, e^{-ik.x} \, + \, a^\dagger(k)\, e^{ik.x}\right]. \tag{3}$$

The commutators are

$$\begin{aligned} \left[a(k), a^\dagger(k')\right] &= 2(2\pi)^3 k^+ \delta^3(k-k'), \\ \left[a(k), a(k')\right] &= \left[a^\dagger(k), a^\dagger(k')\right] = 0. \end{aligned} \tag{4}$$

Single particle state

$$\mid k\rangle = a^\dagger(k) \mid 0\rangle \tag{5}$$

and has the normalization

$$\langle k' \mid k\rangle = 2(2\pi)^3 k^+ \delta^3(k-k'). \tag{6}$$

First let us derive the canonical equal $x^+$ commutation relation for the scalar field. For free field theory, the commutator of $\phi(x)$ and $\phi(y)$ is known for arbitrary $x$ and $y$. We have (see for example Bjorken and Drell (1965)),

$$[\phi(x), \phi(y)] = i\Delta(x-y) \tag{7}$$

where

$$\Delta(x-y) = -i \int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-\mu^2)\epsilon(k^0)e^{-ik.(x-y)}. \tag{8}$$

We have $k^+ = k^0 + k^3$. Thus $\frac{k^+}{k^0} = 1 + \frac{k^3}{k^0} > 0$ on the mass shell and hence $\epsilon(k^0) \rightarrow \epsilon(k^+)$. Thus in terms of light-front variables

$$\begin{aligned}\Delta(x-y) = & -\frac{i}{2}\int \frac{d^2k^\perp}{(2\pi)^3}\int_{-\infty}^{+\infty} dk^+ \int_{-\infty}^{+\infty} dk^- \delta(k^+k^- - (k^\perp)^2 - \mu^2) \\ & \epsilon(k^+)e^{-i(\frac{1}{2}k^-(x^+-y^+)+\frac{1}{2}k^+(x^--y^-)-k^\perp.(x^\perp-y^\perp))}. \end{aligned} \tag{9}$$

From (7) and (9) it is easy to show that

$$[\phi(x), \phi(y)]_{x^+=y^+} = -\frac{i}{4}\epsilon(x^- - y^-)\delta^2(x^\perp - y^\perp) \tag{10}$$

where $\epsilon$ is the antisymmetric step function, $\epsilon(x) = \theta(x) - \theta(-x)$.

The above commutation relation is to be contrasted with the corresponding commutation relation in equal-time theory, namely,

$$[\phi(x), \phi(y)]_{x^0=y^0} = 0. \tag{11}$$

We note that for $x^0 = y^0$, the two fields are separated by a space-like interval, the commutator has to vanish (condition of microscopic causality). For $x^+ = y^+$, if $x^\perp \neq y^\perp$, the two fields are separated by a space-like distance and hence the commutator has to vanish. On the other hand, for $x^+ = y^+$ and $x^\perp = y^\perp$, the two fields are separated by a light-like distance and hence the commutator need not vanish.

Next we consider the scalar field propagator. Let $\bar{S}_B$ denote scalar field propagator in light-front theory. We have

$$\begin{aligned} i\bar{S}_B(x-y) & =< 0 \mid T^+\phi(x)\phi(y) \mid 0 > \\ & = \theta(x^+ - y^+) < 0 \mid \phi(x)\phi(y) \mid 0 > \\ & \quad +\theta(y^+ - x^+) < 0 \mid \phi(y)\phi(x) \mid 0 > . \end{aligned} \tag{12}$$

Using (3) and (4) one can show that

$$\begin{aligned} i\bar{S}_B(x-y) & = \int \frac{d^4k}{(2\pi)^4} \; e^{-ik.(x-y)} \; \frac{i}{k^2 - \mu^2 + i\epsilon} \\ & = iS_B^F(x-y) \end{aligned} \tag{13}$$

where $S_B^F$ is the Feynman propagator for the scalar field. Thus for a scalar field, light-front propagator is the same as the Feynman propagator.

### 2.2 Fermion Field

The equation of motion

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{14}$$

in light-front variables is

$$\left(\frac{i}{2}\gamma^+\partial^- + \frac{i}{2}\gamma^-\partial^+ - i\gamma^\perp.\partial^\perp - m\right)\psi = 0. \tag{15}$$

Define

$$\psi^\pm = \Lambda^\pm \psi, \tag{16}$$

where $\Lambda^\pm = \frac{1}{4}\gamma^\mp\gamma^\pm$.

From (15), it follows that

$$\psi^- = \frac{1}{i\partial^+}(i\alpha^\perp.\partial^\perp + \gamma^0 m)\psi^+. \tag{17}$$

Thus $\psi^-$ is a constrained field since at any $x^+$ it is determined by $\psi^+$. The equation of motion for the dynamical field $\psi^+$ is

$$i\partial^-\psi^+ = \frac{-(\partial^\perp)^2 + m^2}{i\partial^+}\psi^+. \tag{18}$$

Note that the fermion mass appears quadratically in the above equation.

Consider now the equal $x^+$ commutation relation for the dynamical field $\psi^+$. We start from the solution of the free spin-half field theory in equal time:

$$\psi(x,t) = \sum_s \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{m}{E_k}}\left[b(k,s)u(k,s)e^{-ik.x} + d^\dagger(k,s)v(k,s)e^{ik.x}\right]. \tag{19}$$

It follows that (see for example, Bjorken and Drell (1965))

$$\begin{aligned}\{\psi(x,t), \psi^\dagger(y,t')\} &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2E_k} \\ &\qquad \left[(\not{k} + m)\gamma^0 e^{-ik.(x-y)} + (\not{k} - m)\gamma^0 e^{ik.(x-y)}\right] \\ &= (i\not{\partial}_x + m)\gamma^0 i\Delta(x-y).\end{aligned} \tag{20}$$

From the above equation it is easy to show that the equal $x^+$ commutation relation of $\psi^+$ and ${\psi^+}^\dagger$ is

$$\{\psi^+(x), {\psi^+}^\dagger(y)\}_{x^+=y^+} = \Lambda^+\delta(x^- - y^-)\delta^2(x^\perp - y^\perp). \tag{21}$$

Free fermion field operator in light-front theory can be written as (Kogut and Soper (1970), Chang, Root, and Yan (1973))

$$\psi(x) = \int \frac{dk^+ d^2k^\perp}{2k^+(2\pi)^3} \sum_\lambda \left[ b_\lambda(k)\, u_\lambda(k)\, e^{-ik.x} + d^\dagger_\lambda(k)\, v_\lambda(k)\, e^{ik.x} \right] \tag{22}$$

Let $\bar{S}_F$ denote fermion field propagator (Chang and Yan (1973)) in light-front theory.

$$\begin{aligned} i\bar{S}_F(x-y) = &< 0 \mid T^+ \psi(x)\bar{\psi}(y) \mid 0 > \\ = &\ \theta(x^+ - y^+) < 0 \mid \psi(x)\bar{\psi}(y) \mid 0 > \\ &-\theta(y^+ - x^+) < 0 \mid \bar{\psi}(y)\psi(x) \mid 0 > . \end{aligned} \tag{23}$$

Using (22) for the field operator, we can show that the light-front propagator for the fermion field is

$$\begin{aligned} i\bar{S}_F(x-y) &= i \int \frac{d^4k}{(2\pi)^4} \frac{\not{k}_{on} + m}{k^2 - m^2 + i\epsilon} e^{-ik.(x-y)} \\ &= i \int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} \left[ \frac{1}{\not{k} - m + i\epsilon} - \frac{1}{2}\frac{\gamma^+}{k^+} \right] \\ &= iS_F(x-y) - \frac{\gamma^+}{4}\delta(x^+ - y^+)\delta^2(x^\perp - y^\perp)\epsilon(x^- - y^-) \end{aligned} \tag{24}$$

where $S_F$ is the Feynman propagator and $\not{k}_{on} = \frac{1}{2}\gamma^+ \frac{(k^\perp)^2 + m^2}{k^+} + \frac{1}{2}\gamma^- k^+ - \gamma^\perp.k^\perp$. We note that for the fermion field, light-front propagator differs from the Feynman propagator by an instantaneous propagator.

### 2.3 Massless Vector Field

The equation of motion in light-front variables is

$$\partial^+ \left[ \frac{1}{2}\partial^+ A^- + \frac{1}{2}\partial^- A^+ - \partial^\perp . A^\perp \right] - \left( \partial^+\partial^- - {\partial^\perp}^2 \right) A^+ = 0, \tag{25}$$

$$\partial^i \left[ \frac{1}{2}\partial^+ A^- + \frac{1}{2}\partial^- A^+ - \partial^\perp . A^\perp \right] - \left( \partial^+\partial^- - {\partial^\perp}^2 \right) A^i = 0, \tag{26}$$

$$\partial^- \left[ \frac{1}{2}\partial^+ A^- + \frac{1}{2}\partial^- A^+ - \partial^\perp . A^\perp \right] - \left( \partial^+\partial^- - {\partial^\perp}^2 \right) A^- = 0. \tag{27}$$

Choose the gauge (Kogut and Soper (1970), Neville and Rohrlich (1971))

$$A^+ = 0. \tag{28}$$

This gauge choice is known as infinite-momentum gauge, null-plane gauge, light-cone gauge and light-front gauge. From (25), we have

$$\partial^+ A^- = 2\partial^\perp.A^\perp \quad + F(x^+, x^\perp) \tag{29}$$

Thus $A^-$ is not a dynamical variable. Choosing $F$ to be zero, the dynamical variables $A^i$ obey massless Klein-Gordon equation.

Since the dynamical variable $A^i$ obey massless Klein-Gordon equation, we can follow the same route we have taken for the free scalar field and write the field operator in quantum theory as

$$A^j(x) = \int \frac{dk^+ d^2k^\perp}{2k^+(2\pi)^3} \sum_\lambda \delta_{j\lambda} \left[ a_\lambda(k) e^{-ik.x} \ + \ a^\dagger_\lambda(k) e^{ik.x} \right] \tag{30}$$

with

$$\begin{aligned} \left[ a_\lambda(k), a^\dagger_\sigma(k') \right] \ &= \ 2(2\pi)^3 k^+ \delta_{\lambda\sigma} \delta^3(k-k'), \\ \left[ a_\lambda(k), a_\sigma(k') \right] \ &= 0, \ \left[ a^\dagger_\lambda(k), a^\dagger_\sigma(k') \right] = 0. \end{aligned} \tag{31}$$

The equal $x^+$ commutation relation is

$$\left[ A^i(x), A^j(y) \right]_{x^+=y^+} = \frac{-i}{4} \delta_{ij} \ \epsilon(x^- - y^-) \ \delta^2(x^\perp - y^\perp). \tag{32}$$

With $F = 0$, we have,

$$A^-(x^-, x^\perp) = \frac{1}{2} \int dy^- \epsilon(x^- - y^-) \partial^i A^i(y^-, x^\perp). \tag{33}$$

Explicitly, using (30), we have,

$$A^-(x) = \int \frac{dk^+ d^2k^\perp}{2k^+(2\pi)^3} \sum_\lambda \delta_{j\lambda} \frac{2k^j}{k^+} \left[ a_\lambda(k) e^{-ik.x} \quad + \quad a^\dagger_\lambda(k) e^{ik.x} \right]. \tag{34}$$

Introducing the polarization vectors

$$\epsilon^\mu_1(k) = \frac{1}{k^+}(0, 2k^1, k^+, 0), \ \ \epsilon^\mu_2(k) = \frac{1}{k^+}(0, 2k^2, 0, k^+), \tag{35}$$

we can write

$$A^\mu(x) = \int \frac{dk^+ d^2k^\perp}{2k^+(2\pi)^3} \sum_\lambda \epsilon^\mu_\lambda(k) \left[ a_\lambda(k) e^{-ik.x} \ + \ a^\dagger_\lambda(k) e^{ik.x} \right]. \tag{36}$$

Note that,

$$\partial_\mu A^\mu = 0. \tag{37}$$

Introducing the four-vector $\eta = (0, 2, 0^\perp)$ we have the relation

$$\sum_\lambda \epsilon^\mu_\lambda(k)\epsilon^\nu_\lambda(k) = -g^{\mu\nu} + \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{k^+} - \eta^\mu\eta^\nu \frac{k^2}{(k^+)^2}. \tag{38}$$

Let $\bar{S}_V$ denote the massless vector field propagator (Yan (1973b)) in light-front theory. We have

$$\begin{aligned} i(\bar{S}_V)^{\mu\nu}(x-y) &= \langle 0 \mid T^+ A^\mu(x) A^\nu(y) \mid 0\rangle \\ &= \theta(x^+ - y^+)\langle 0 \mid A^\mu(x)A^\nu(y) \mid 0\rangle \\ &\quad + \theta(y^+ - x^+)\langle 0 \mid A^\nu(y)A^\mu(x) \mid 0\rangle. \end{aligned} \tag{39}$$

Using the expansion (36) we have

$$\begin{aligned} i(\bar{S}_V)^{\mu\nu}(x-y) = &\int \frac{d^4k}{(2\pi)^4} e^{-ik.(x-y)} \frac{i}{k^2 + i\epsilon} \\ &\left[-g^{\mu\nu} + \frac{\eta^\mu k^\nu + \eta^\nu k^\mu}{k^+} - \eta^\mu\eta^\nu \frac{k^2}{(k^+)^2}\right]. \end{aligned} \tag{40}$$

## 3 Poincare Generators and Algebra

### 3.1 Lorentz Group

Let us first consider a pure boost along the negative 3-axis. The relationship between space and time of two systems of coordinates, one $\tilde{S}$ in uniform motion along the negative 3-axis with speed $v$ relative to other $S$ is given by $\tilde{x}^0 = \gamma(x^0 + \beta x^3)$, $\tilde{x}^3 = \gamma(x^3 + \beta x^0)$, with $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$. Introduce the parameter $\phi$ such that $\gamma = \cosh\phi$, $\beta\gamma = \sinh\phi$. In terms of the light-front variables,

$$\tilde{x}^+ = e^\phi x^+,\ \tilde{x}^- = e^{-\phi} x^-. \tag{41}$$

Thus boost along the 3-axis becomes a scale transformation for the variables $\tilde{x}^+$ and $\tilde{x}^-$ and $x^+ = 0$ is invariant under boost along the 3-axis.

Let us denote the three generators of boosts by $K^i$ and the three generators of rotations by $J^i$ in equal-time dynamics. Define $E^1 = -K^1 + J^2$, $E^2 = -K^2 - J^1$, $F^1 = -K^1 - J^2$, and $F^2 = -K^2 + J^1$. The explicit expressions for the 6 generators $K^3$, $E^1$, $E^2$, $J^3$, $F^1$, and $F^2$ in the finite dimensional representation using the conventions of Ryder (1985) are

$$K^3 = -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ E^1 = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$E^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ J^3 = -i \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F^1 = -i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \ F^2 = -i \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Note that $K^3$, $E^1$, $E^2$, and $J^3$ leave $x^+ = 0$ invariant and are kinematical generators while $F^1$ and $F^2$ do not and are dynamical generators.

It follows that

$$[F^1, F^2] = 0, [J^3, F^i] = i\epsilon^{ij} F^j. \tag{42}$$

Thus $J^3$, $F^1$ and $F^2$ form a closed algebra. Also

$$[E^1, E^2] = 0, [K^3, E^i] = iE^i. \tag{43}$$

Thus $K^3$, $E^1$ and $E^2$ also form a closed algebra.

### 3.2 Algebra

From the Lagrangian density one may construct the stress tensor $T^{\mu\nu}$ and from the stress tensor one may construct a four-momentum $P^\mu$ and a generalized angular momentum $M^{\mu\nu}$.

$$P^\mu = \frac{1}{2} \int dx^- d^2x^\perp \, T^{+\mu}, \tag{44}$$

$$M^{\mu\nu} = \frac{1}{2} \int dx^- d^2x^\perp [x^\nu \, T^{+\mu} - x^\mu \, T^{+\nu}]. \tag{45}$$

Note that $M^{\mu\nu}$ is antisymmetric and hence has six independent components. Poincare algebra in terms of $P^\mu$ and $M^{\mu\nu}$ is (see for example, Ryder (1985))

$$[P^\mu, P^\nu] = 0, \tag{46}$$

$$[P^\mu, M^{\rho\sigma}] = i[g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho], \tag{47}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i[-g^{\mu\rho} M^{\nu\sigma} + g^{\mu\sigma} M^{\nu\rho} - g^{\nu\sigma} M^{\mu\rho} + g^{\nu\rho} M^{\mu\sigma}]. \tag{48}$$

In light-front dynamics $P^-$ is the Hamiltonian and $P^+$ and $P^i$ $(i = 1, 2)$ are the momenta. $M^{+-} = 2K^3$ and $M^{+i} = E^i$ are the boosts. $M^{12} = J^3$ and $M^{-i} = F^i$ are the rotations. The following table summarizes the commutation relations between the Poincare generators in light-front dynamics.

| | $P^+$ | $P^1$ | $P^2$ | $K^3$ | $E^1$ | $E^2$ | $J^3$ | $F^1$ | $F^2$ | $P^-$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $P^+$ | 0 | 0 | 0 | $-iP^+$ | 0 | 0 | 0 | $2iP^1$ | $2iP^2$ | 0 |
| $P^1$ | 0 | 0 | 0 | 0 | $iP^+$ | 0 | $-iP^2$ | $iP^-$ | 0 | 0 |
| $P^2$ | 0 | 0 | 0 | 0 | 0 | $-iP^+$ | $iP^1$ | 0 | $iP^-$ | 0 |
| $K^3$ | $iP^+$ | 0 | 0 | 0 | $iE^1$ | $iE^2$ | 0 | $-iF^1$ | $-iF^2$ | $-iP^-$ |
| $E^1$ | 0 | $-iP^+$ | 0 | $-iE^1$ | 0 | 0 | $-iE^2$ | $-2iK^3$ | $-2iJ^3$ | $-2iP^1$ |
| $E^2$ | 0 | 0 | $-iP^+$ | $-iE^2$ | 0 | 0 | $iE^1$ | $2iJ^3$ | $2iK^3$ | $-2iP^2$ |
| $J^3$ | 0 | $iP^2$ | $-iP^1$ | 0 | $iE^2$ | $-iE^1$ | 0 | $iF^2$ | $-iF^1$ | 0 |
| $F^1$ | $-2iP^1$ | $-iP^-$ | 0 | $iF^1$ | $-2iK^3$ | $-2iJ^3$ | $-iF^2$ | 0 | 0 | 0 |
| $F^2$ | $-2iP^2$ | 0 | $-iP^-$ | $iF^2$ | $2iJ^3$ | $-2iK^3$ | $iF^1$ | 0 | 0 | 0 |
| $P^-$ | 0 | 0 | 0 | $iP^-$ | $2iP^1$ | $2iP^2$ | 0 | 0 | 0 | 0 |

### 3.3 Free Scalar Field: Generators in Fock Representation

In this section, as an example, we explicitly construct the Poincare generators of free scalar field theory in Fock representation (Flory (1970)).

From the Lagrangian density, we obtain the conserved symmetric stress tensor. The stress tensor

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}. \tag{49}$$

with

$$\mathcal{L} = \frac{1}{2}\partial_\sigma \phi \partial^\sigma \phi - \frac{1}{2}\mu^2 \phi^2. \tag{50}$$

The momentum operators are given by

$$P^+ = \frac{1}{2}\int dx^- d^2x^\perp \, \partial^+ \phi \partial^+ \phi. \tag{51}$$

$$P^i = \frac{1}{2}\int dx^- d^2x^\perp \, \partial^+ \phi \partial^i \phi. \tag{52}$$

The Hamiltonian operator

$$P^- = \frac{1}{2}\int dx^- d^2x^\perp \, \left[\partial^i \phi \partial^i \phi + \mu^2 \phi^2\right]. \tag{53}$$

The generators of boosts are (at $x^+ = 0$),

$$K^3 = \frac{1}{4}\int dx^- d^2x^\perp \, x^- \partial^+ \phi \partial^+ \phi, \tag{54}$$

and

$$E^i = \frac{1}{2} \int dx^- d^2x^\perp \, x^i \, \partial^+ \phi \partial^+ \phi. \tag{55}$$

The generators of rotations are

$$J^3 = -\frac{1}{2} \int dx^- d^2x^\perp \, \partial^+ \phi \, \left[ x^1 \partial^2 \phi \, - \, x^2 \partial^1 \phi \right] \tag{56}$$

and

$$F^i = -\frac{1}{2} \int dx^- d^2x^\perp \, \left[ x^- \partial^+ \phi \partial^i \phi - x^i (\partial^\perp \phi . \partial^\perp \phi + \mu^2 \phi^2) \right]. \tag{57}$$

In terms of Fock space operators, we have,

$$P^+ = \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} k^+ a^\dagger(k) a(k). \tag{58}$$

$$P^i = \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} k^i a^\dagger(k) a(k). \tag{59}$$

$$P^- = \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} \frac{\mu^2 + (k^\perp)^2}{k^+} a^\dagger(k) a(k). \tag{60}$$

$$K^3 = i \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} \left( \frac{\partial}{\partial k^+} a^\dagger(k) \right) k^+ a(k). \tag{61}$$

$$E^i = -i \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} \left( \frac{\partial}{\partial k^i} a^\dagger(k) \right) k^+ a(k). \tag{62}$$

$$J^3 = -i \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} \left( [k^1 \frac{\partial}{\partial k^2} - k^2 \frac{\partial}{\partial k^1}] a^\dagger(k) \right) a(k). \tag{63}$$

$$\begin{aligned} F^i \; &= -i \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} \frac{\mu^2 + {k^\perp}^2}{k^+} \left( \frac{\partial}{\partial k^i} a^\dagger(k) \right) a(k) \\ &-2i \int \frac{dk^+ d^2k^\perp}{2k^+ (2\pi)^3} k^i \left( \frac{\partial a^\dagger(k)}{\partial k^+} \right) a(k). \end{aligned} \tag{64}$$

For a single particle, we have,

$$
\begin{aligned}
P^+ \mid p\rangle &= p^+ \mid p\rangle, && (65)\\
P^i \mid p\rangle &= p^i \mid p\rangle, && (66)\\
P^- \mid p\rangle &= \frac{(p^\perp)^2 + \mu^2}{p^+} \mid p\rangle, && (67)\\
K^3 \mid p\rangle &= ip^+ \frac{\partial}{\partial p^+} \mid p\rangle, && (68)\\
E^i \mid p\rangle &= -ip^+ \frac{\partial}{\partial p^i} \mid p\rangle, && (69)\\
J^3 \mid p\rangle &= i\left[p^2 \frac{\partial}{\partial p^1} - p^1 \frac{\partial}{\partial p^2}\right] \mid p\rangle, && (70)\\
F^i \mid p\rangle &= -\big[i\frac{(p^\perp)^2 + \mu^2}{p^+} \frac{\partial}{\partial p^i} + 2ip^i \frac{\partial}{\partial p^+}\big] \mid p\rangle. && (71)
\end{aligned}
$$

## 4 Two-Dimensional Quantum Electrodynamics

### 4.1 Introduction

In this lecture we discuss two dimensional (one space-one time) Quantum Electrodynamics (QED) in light-front dynamics. Our main purpose is to exhibit some of the simple features of relativistic bound states in the simplest setting. We also discuss some aspects of renormalization and anomaly.

We study the bound state dynamics of $\mathrm{QED}_2$ in the truncated space of one fermion-anti fermion pair. In this model, with the gauge choice $A^+ = 0$ on the light-front we have fermions and antifermions interacting via instantaneous interactions. It turns out that just with one pair we have a reasonably good description of the ground state in both weak coupling (non-relativistic) and strong coupling (relativistic) domains.

Just for notational convenience we omit the superscript + for longitudinal momenta in this section.

### 4.2 Hamiltonian

The Lagrangian density for QED is given by

$$
\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \tag{72}
$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $D^\mu = \partial^\mu + ieA^\mu$. We pick the light-front gauge $A^+ = 0$. From the equations of motion

$$
(i\not{D} - m)\psi = 0, \tag{73}
$$

$$
\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi, \qquad . \tag{74}
$$

we get the constraint equations

$$\psi^- = \frac{1}{i\partial^+}\gamma^0 m\psi^+, \tag{75}$$

$$A^- = -\frac{4e}{(\partial^+)^2}\psi^{+\dagger}\psi^+, \qquad . \tag{76}$$

The equation of motion for the dynamical variable $\psi^+$ is

$$i\partial^-\psi^+ = m^2\frac{1}{i\partial^+}\psi^+ - \left[4e^2\frac{1}{(\partial^+)^2}\left(\psi^{+\dagger}\psi^+\right)\right]\psi^+. \tag{77}$$

The symmetric energy momentum tensor is

$$\begin{aligned} T^{\mu\nu} &= -F^{\mu\lambda}F^{\nu}_{\lambda} + \frac{1}{2}\bar{\psi}\left(\gamma^\mu D^\nu + \gamma^\nu D^\mu\right)\psi \\ &\quad -g^{\mu\nu}\left(-\frac{1}{4}F^{\lambda\sigma}F_{\lambda\sigma} + \bar{\psi}(i\not{D} - m)\psi\right). \end{aligned} \tag{78}$$

In the gauge $A^+ = 0$, the momentum

$$P^+ = \frac{1}{2}\int dx^- 2i\psi^{+\dagger}\partial^+\psi^+ \tag{79}$$

and the Hamiltonian is given by

$$P^- = \int dx^-\left(m^2\psi^{+\dagger}\frac{1}{i\partial^+}\psi^+ - 2e^2\psi^{+\dagger}\psi^+\frac{1}{(\partial^+)^2}\psi^{+\dagger}\psi^+\right). \tag{80}$$

Note that the Hamiltonian has only fermion degrees of freedom which drastically simplifies Fock space structure. In the following first we truncate the Fock space to a fermion-antifermion pair. We give the relevant terms in the Hamiltonian also in terms of Fock space operators.

By projecting the eigenvalue equation

$$P^+P^- \mid \Psi\rangle = M^2 \mid \Psi\rangle \tag{81}$$

on to a pair of free states, we arrive at the bound state equation in QED. The bound state equation is shown to reproduce the well-known results for the ground state in the massless (ultra-relativistic) limit. The bound state equation is also shown to reproduce the well-known results in the heavy mass (non-relativistic) limit.

### 4.3 Bound State Equation in QED

The field operator is

$$\psi^+(x) = \int \frac{dk}{4\pi\sqrt{k}} \left[ b(k) e^{ik.x} + d^\dagger(k) e^{-ik.x} \right] \tag{82}$$

with

$$\left\{ b(k), b^\dagger(k') \right\} = \left\{ d(k), d^\dagger(k') \right\} = 4\pi k \delta(k-k') \; . \tag{83}$$

The relevant terms in the Hamiltonian are

$$P^- = P^-_{\text{free}} + P^-_{\text{int}} \tag{84}$$

where

$$\begin{aligned} P^-_{\text{free}} &= \int \frac{dk}{4\pi k} \left[ b^\dagger(k) b(k) + d^\dagger(k) d(k) \right] \\ &\quad \times \left[ \frac{m^2}{k} + 2e^2 \int \frac{dk_1}{4\pi} \left( \frac{1}{(k-k_1)^2} - \frac{1}{(k+k_1)^2} \right) \right] \; , \end{aligned} \tag{85}$$

$$\begin{aligned} P^-_{\text{int}} &= -4e^2 \int \frac{dk_1}{4\pi\sqrt{k_1}} \int \frac{dk_2}{4\pi\sqrt{k_2}} \int \frac{dk_3}{4\pi\sqrt{k_3}} \int \frac{dk_4}{4\pi\sqrt{k_4}} 4\pi\delta(k_1-k_2-k_3+k_4) \\ &\quad \times b^\dagger(k_1) b(k_2) d^\dagger(k_4) d(k_3) \left[ \frac{1}{(k_1-k_2)^2} - \frac{1}{(k_1+k_4)^2} \right] . \end{aligned} \tag{86}$$

Note that we have generated self-energy contributions to the mass (85) by normal ordering the instantaneous four-fermion interaction.

We expand the state vector $\mid \Psi >$ in terms Fock space states and truncate to a fermion-antifermion pair:

$$\begin{aligned} \mid \Psi(P) > &= \int \frac{dp_1}{\sqrt{4\pi p_1}} \frac{dp_2}{\sqrt{4\pi p_2}} \phi_2(p_1,p_2) b^\dagger(p_1) d^\dagger(p_2) \mid 0 > \\ &\quad \times \sqrt{2(2\pi)P}\,\delta(P-p_1-p_2). \end{aligned} \tag{87}$$

By projecting the eigenvalue equation (81) on to a pair of free states and introducing the momentum fraction variables ($x = \frac{p_1}{P}$, $\phi_2(p_1,p_2) = \frac{1}{\sqrt{P}}\psi_2(x)$ etc. ) we arrive at the bound state equation

$$M^2 \psi_2(x) = \frac{m^2}{x(1-x)} \psi_2(x) - \frac{e^2}{\pi} \int dy \frac{\psi_2(y) - \psi_2(x)}{(x-y)^2} + \frac{e^2}{\pi} \int_0^1 dy \psi_2(y) \tag{88}$$

The factor proportional to $\psi(x)$ in the third term is the self-energy contribution.

### 4.4 Relativistic Limit

The bound state equation (88) would have exhibited severe $\frac{1}{x^2}$ divergences coming from the instantaneous gauge boson exchange if self-energy contributions were ignored. Such divergences are present in the eigenvalue equation for single fermion. A detailed and excellent discussion of these divergences and corresponding regulators in the context of confinement and asymptotic freedom in $QCD_2$ can be found in Callan, Coote and Gross (1976) and Einhorn (1976).

In the extreme relativistic limit ($m \to 0$), (88) shows that $\psi_2 = \theta(x)\theta(1-x)$ is a solution with eigenvalue $M^2 = \frac{e^2}{\pi}$. This is the well-known Schwinger result in two-dimensional massless electrodynamics (Schwinger model).

The result that a single fermion-antifermion pair reproduces the well-known result in the extreme strong coupling limit in light-front quantization is in fact nontrivial. In equal-time quantization, in $A^3 = 0$ gauge for example, restriction to a single pair is a valid approximation only in the extreme nonrelativistic limit. For a comparison of bound state equations in equal-time and light-front cases in the context of $QCD_2$ see Hanson, Peccei, and Prasad (1976).

### 4.5 Nonrelativistic Limit

In the nonrelativistic limit (fermion mass $\to \infty$) , the last term in (88) which corresponds to the "annihilation channel" can be ignored. Then the bound state equations for QED and QCD are identical except for a rescaling of the coupling constant. Let us start from (88) without the last term.

$$M^2\psi_2(x_1) - \frac{m^2}{x_1(1-x_1)}\psi_2(x_1) + \frac{e^2}{\pi}\int dy_1 \frac{\psi_2(y_1) - \psi_2(x_1)}{(x_1 - y_1)^2} = 0. \tag{89}$$

Introduce the variable $q$ via

$$x_1 = \frac{1}{2}\left(1 - \frac{q}{\epsilon(q)}\right) \tag{90}$$

with $\epsilon(q) = \sqrt{q^2 + m^2}$. Note that the range of $q$ is $-\infty < q < +\infty$. Utilizing the fact that $\epsilon \approx m$,

$$(x_1 - y_1)^2 \approx \frac{(q - q')^2}{4m^2}. \tag{91}$$

Introducing $\bar{B} = B/m - \frac{B^2}{4m^2}$ where $B = 2m - M$, we have,

$$[\bar{B} + q^2]\psi(q) = \frac{e^2}{2\pi} P \int dq' \frac{\psi(q') - \psi(q)}{(q - q')^2}. \tag{92}$$

The second term on r.h.s. is the self-energy correction which also vanishes in the nonrelativistic limit.

The Fourier transform of $- \mid z \mid \psi(z)$ leads to $\frac{1}{2\pi} P \int dq' \frac{\psi(q')}{(q-q')^2}$, and we arrive at the coordinate space equation

$$\left[ -\frac{\partial^2}{\partial u^2} + \mid u \mid \right] \psi(u) = (-) \lambda \psi(u) \tag{93}$$

where $u = e^{\frac{2}{3}} z$ and $\lambda = \bar{B} e^{-\frac{4}{3}}$. The solution to (93) are the well known Airy functions. A discussion of (93) is given by Hamer (1977).

### 4.6 Anomaly

In this subsection we follow the discussion in Bergknoff (1977). Classically, in the massless limit, chiral symmetry of the $QED_2$ Lagrangian leads to the conservation of axial vector current $j_5^\mu = \bar{\psi} \gamma^\mu \gamma_5 \psi$, $\partial_\mu j_5^\mu = 0$. Let us calculate the divergence of the axial vector current in the quantum theory.

We have

$$\partial_\mu j_5^\mu = \frac{1}{2} \partial^+ j_5^- + \frac{1}{2} \partial^- j_5^+ . \tag{94}$$

In one space - one time dimensions, the vector current $j^\mu = \bar{\psi} \gamma^\mu \psi$ and the axial vector current $j_5^\mu$ are related by

$$j_5^\mu = -\epsilon^{\mu\nu} j_\nu , \tag{95}$$

where $\epsilon^{\mu\nu}$ is the antisymmetric tensor, $\epsilon^{+-} = -2$. Thus

$$j_5^+ = j^+ \text{ and } j_5^- = -j^- . \tag{96}$$

From the conservation of the vector current $j^\mu$, we have

$$\partial^+ j^- = -\partial^- j^+ \tag{97}$$

Thus

$$\partial_\mu j_5^\mu = \partial^- j^+ = -i \left[ j^+ , P^- \right] . \tag{98}$$

Thus we need to calculate the commutator of the plus component of the vector current and the Hamiltonian. This evaluation is most easily carried out in momentum space utilizing the Fourier mode expansion of the field $\psi^+$.

In the massless limit, the Hamiltonian can be written as

$$P^- = \frac{e^2}{8\pi} \int_{-\infty}^{+\infty} \frac{dp}{(p)^2} \tilde{j}^+(p) \tilde{j}^+(-p), \tag{99}$$

where we have introduced the Fourier transform of the current,

$$j^+(x) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} dp e^{i \frac{1}{2} p x^-} \tilde{j}^+(p). \tag{100}$$

Thus we need to calculate the commutator of the plus component of the currents, $\left[\tilde{j}^{+}(p), \tilde{j}^{+}(q)\right]$.

Using the Fourier mode expansion of the field (82), it is easily shown that,

$$\langle 0 \mid \left[j^{+}(x), j^{+}(y)\right] \mid 0\rangle \ = 4 \int_{0}^{\infty} \frac{dk_1}{4\pi} \int_{0}^{\infty} \frac{dk_2}{4\pi} \left[e^{-i\frac{1}{2}(k_1+k_2)(x^{-}-y^{-})} - c.c.\right]. \tag{101}$$

Thus, we have,

$$\langle 0 \mid \left[\tilde{j}^{+}(p), \tilde{j}^{+}(q)\right] \mid 0\rangle = 4q\delta(p+q). \tag{102}$$

In the absence of any q-number structure, we have, $\left[\tilde{j}^{+}(p), \tilde{j}^{+}(q)\right] = 4q\delta(p+q)$. An explicit evaluation, then, leads to

$$\left[\tilde{j}^{+}(p), P^{-}\right] = -\frac{e^2}{\pi} \frac{\tilde{j}^{+}(p)}{p}. \tag{103}$$

From (98) we have

$$\frac{\partial}{\partial x^{+}} \tilde{j}^{+} = i \frac{e^2}{2\pi} \frac{\tilde{j}^{+}(p)}{p} \tag{104}$$

which shows that $\partial_\mu j_5^\mu$ is not zero. In position space the above equation leads to

$$\frac{\partial^2 j^{+}(x)}{\partial x^{+} \partial x^{-}} = -\frac{e^2}{4\pi} j^{+}. \tag{105}$$

Thus we see that (1) in the quantum theory, divergence of the axial vector current is nonzero, even though it is zero in the classical theory, (2) $j^{+}$ obeys the Klein-Gordon equation for a massive scalar field with $m^2 = \frac{e^2}{\pi}$.

## 5 Light-Front Power Counting and its Consequences

In this section we discuss the light-front power counting introduced by Wilson (Wilson (1990), Wilson et al. (1994)). To illustrate its consequences in a simple example in one plus one dimensions we first discuss the massive Thirring model. Then we discuss the motivation for light-front power counting and give the power assignments for dynamical variables and the Hamiltonian in three plus one dimensions. Simple examples of Hamiltonians involving scalars and fermions are given at the tree level. Appealing to power counting alone leads to a large number of free parameters in the theory. The idea of reducing the number of free parameters by implementing the symmetries is illustrated using a simple example in Yukawa theory.

### 5.1 Massive Thirring Model

Power counting is different in light-front dynamics. For example, in two dimensions, $\psi^+$ has no mass dimension whereas in equal-time theory $\psi$ has mass dimension $\frac{1}{2}$. In both cases the scalar field $\phi$ has no mass dimension. Thus in light-front theory in one plus dimensions infinite number of terms are possible in the interaction. However, in two-dimensional gauge theories and two-dimensional Yukawa model, the coupling constant ($e$ and $g$ respectively) has the dimension of mass. By dimensional analysis, the Hamiltonian $P^-$ has dimension two in units of mass. Accordingly, in gauge theory case the highest power of coupling allowed by power counting is $e^2$ and in Yukawa model highest powers of coupling allowed are $g$ (must be accompanied by a mass $m$ to balance dimensions) and $g^2$. Explicit construction of the canonical light-front Hamiltonian in these cases shows that the interaction terms obey these power counting rules.

If the coupling $g^2$ is dimensionless infinite number of terms appear in $P^-$ for theories in two dimensions. In equal-time theory, four-fermion interactions have dimensionless coupling constant. Since $\psi$ carry mass dimension $\frac{1}{2}$, six-fermion interactions etc. are not allowed by power counting. On other hand, in light-front theory $\psi^+$ carry no mass dimension, and hence infinite number of terms are allowed for fermionic interactions in $P^-$ by power counting just like bosonic interactions in equal-time theory in one plus one dimensions. By dimensional arguments a constant with dimensions of $m^2$ has to appear as a overall multiplicative factor in front of the interaction Hamiltonian. In the following we illustrate these features in the context of massive Thirring model.

The Lagrangian density for massive Thirring model is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{2}g^2(\bar{\psi}\gamma^\mu\psi)^2. \tag{106}$$

The equation of motion is

$$i\partial^-\psi^+ = m\gamma^0\psi^- + 2g^2{\psi^-}^\dagger\psi^-\psi^+. \tag{107}$$

To get the *true* equation of motion, we have to eliminate the constraint variable $\psi^-$ which obeys the equation of constraint:

$$i\partial^+\psi^- = m\gamma^0\psi^+ + 2g^2{\psi^+}^\dagger\psi^+\psi^-. \tag{108}$$

As was mentioned before, the equation of constraint is nonlinear, in contrast to the situation in gauge theories and Yukawa model.

The Hamiltonian density is

$$\begin{aligned}\mathcal{H} &= -i{\psi^-}^\dagger\partial^+\psi^- + m\left[{\psi^+}^\dagger\gamma^0\psi^- + {\psi^-}^\dagger\gamma^0\psi^+\right] + 2g^2{\psi^+}^\dagger\psi^+{\psi^-}^\dagger\psi^-. \\ &= m{\psi^+}^\dagger\gamma^0\psi^-. \end{aligned} \tag{109}$$

In order to express the Hamiltonian in terms of the physical degree of freedom $\psi^+$, we need to solve the constraint equation (108).

Following Domokos (1971), introduce the Green function

$$G(x^-, y^-) = \frac{1}{4i}\epsilon(x^- - y^-)e^{-ig^2\left[B(x^-)-B(y^-)\right]}, \tag{110}$$

where

$$B(x^-) = \frac{1}{2}\int dz^- \epsilon(x^- - z^-){\psi^+}^\dagger(z^-)\psi^+(z^-). \tag{111}$$

One can easily verify that

$$\psi^-(x^-) = m\gamma^0 \int dy^- G(x^-, y^-)\psi^+(y^-) \tag{112}$$

satisfies the constraint equation (108). Thus the constraint equation is explicitly solved using the above ansatz.

The Hamiltonian

$$P^- = m^2 \int dx^- \int dy^- {\psi^+}^\dagger(x^-)G(x^-, y^-)\psi^+(y^-). \tag{113}$$

Thus we see explicitly that (1) there are infinite number of terms in the Hamiltonian (which, in this particular case, exponentiates resulting in a closed form) and (2) $m^2$ appears as an overall multiplicative factor. For $g^2 = 0$ we reproduce the free field theory result.

### 5.2 Light-Front Power Counting: Motivation

In conventional Lagrangian field theory, one starts with the terms allowed by power counting in the Lagrangian density. Power counting alone may lead to a large number of arbitrary parameters in the theory. When restrictions from Lorentz invariance and gauge invariance (in the case of gauge theories) are imposed, this number is drastically reduced. By analyzing arbitrary orders of perturbation theory, one discovers that the counterterms are all of the form as the canonical ones, provided the cutoffs respect the imposed symmetries. Following the same path, in QCD for example, we need to construct the most general form (including the canonical terms and counterterms) of the light-front Hamiltonian for QCD. In our case, we have to use the light-front power counting to construct the Hamiltonian. Further, to reduce the number of arbitrary parameters we can impose light-front symmetries.

Why light-front power counting is different? Light-front power counting is in terms of the longitudinal coordinate $x^-$ and the transverse coordinate $x^\perp$. It has been noticed that $x^-$ and $x^\perp$ have to be treated differently. We may give three reasons for doing so: (1) The energy $k^-$ scales differently with $x^-$ and $x^\perp$ scaling. *i.e.,* from the free particle dispersion relation $k^- = \frac{(k^\perp)^2+m^2}{k^+}$, $k^-$ scales as $x^-$ (both are the minus component of four-vectors) and $k^-$ scales as $\frac{1}{(x^\perp)^2}$. (2) $x^-$ does not carry inverse mass dimension, only $x^\perp$ does. (3) Longitudinal scale transformation is operationally identical to the longitudinal boost transformation which is a Lorentz symmetry.

### 5.3 Canonical Power Assignments

Analysis of the canonical light-front Hamiltonian shows that indeed it scales differently under $x^-$ and $x^\perp$ scaling. To determine the scaling properties of the Hamiltonian, first we need to determine the scale dimensions of the dynamical variables (scalar field $\phi$, the plus component of the fermion field $\psi^+$, the transverse component of the gauge field, $A^\perp$, etc.). From the scaling analysis of canonical commutation relations (Wilson et al. (1994)), the power assignments are

$$\begin{aligned} \phi &: \frac{1}{x^\perp} \\ A^\perp &: \frac{1}{x^\perp} \\ \psi^+ &: \frac{1}{\sqrt{x^-}x^\perp}. \end{aligned} \tag{114}$$

The power assignments for the derivatives are

$$\begin{aligned} \partial^\perp &: \frac{1}{x^\perp} \\ \partial^+ &: \frac{1}{x^-}. \end{aligned} \tag{115}$$

Since $\partial^\perp$ carry mass dimension $\frac{1}{\partial^\perp}$ is not allowed in the canonical Hamiltonian whereas $\partial^+$ do not carry mass dimension and hence inverse powers of $\partial^+$ are allowed in the canonical Hamiltonian. The interaction Hamiltonian density $\mathcal{H}$ has the power assignment $\frac{1}{(x^\perp)^4}$. The Hamiltonian does not have a unique scaling behavior in the transverse plane when parameters with dimensions of the mass are present whereas longitudinal scaling behavior is unaffected by mass parameters. For dimensional analysis we assign

$$\begin{aligned} \mathcal{H} &: \frac{1}{(x^\perp)^4} \\ H &: \frac{x^-}{(x^\perp)^2}. \end{aligned} \tag{116}$$

Let us consider some examples of canonical Hamiltonians constructed using the power counting rules.

**Scalar Theory.** Since the power assignment for the scalar field is $\phi : \frac{1}{x^\perp}$, the allowed terms are $\mu^2\phi^2$, $\partial^\perp\phi.\partial^\perp\phi$, $c\phi^3$, and $\phi^4$ where $\mu$ and $c$ have mass dimension. Hence the most general form of the canonical Hamiltonian for the scalar field is

$$\mathcal{H} = c_1\partial^\perp\phi.\partial^\perp\phi + c_2\mu^2\phi^2 + c_3\phi^3 + c_4\phi^4, \tag{117}$$

where $c_1$, $c_2$, and $c_4$ are dimensionless and $c_3$ has mass dimension.

**Fermions Interacting with Scalar (Yukawa Model).** Let us first consider the interaction free parts of the Hamiltonian density. Since the dynamical fermion field $\psi^+$ has the power assignment $\psi^+ : \frac{1}{\sqrt{x^- x^\perp}}$ and the Hamiltonian density has the power assignment $\mathcal{H} = \frac{1}{(x^\perp)^4}$, the inverse longitudinal derivative occurs in the free parts to balance longitudinal scale dimensions. The allowed free parts are ${\psi^+}^\dagger \frac{(\partial^\perp)^2}{\partial^+}\psi^+$, $m^2 {\psi^+}^\dagger \frac{1}{\partial^+}\psi^+$ where $m$ is a mass parameter. The interaction terms allowed are ${\psi^+}^\dagger \phi \frac{1}{\partial^+}\psi^+$, ${\psi^+}^\dagger \frac{1}{\partial^+}(\phi\psi^+)$, ${\psi^+}^\dagger \phi \frac{\gamma^\perp.\partial^\perp}{\partial^+}\psi^+$, ${\psi^+}^\dagger \frac{\gamma^\perp.\partial^\perp}{\partial^+}(\phi\psi^+)$ and ${\psi^+}^\dagger \phi \frac{1}{\partial^+}(\phi\psi^+)$. The presence of nonlocal two fermion - two boson interaction is a consequence of light-front power counting. Note that in this catalogue we have ignored terms which appear as surface terms in the Hamiltonian. By adding the terms for the scalar field Hamiltonian density given in the previous section, we get the most general form of the canonical Hamiltonian density allowed by power counting.

$$
\begin{aligned}
\mathcal{H}_{\rm pc} \; &= c_1 \partial^\perp\phi.\partial^\perp\phi + c_2\mu^2\phi^2 + c_3\phi^3 + c_4\phi^4 \\
&\quad + c_5 {\psi^+}^\dagger \frac{(\partial^\perp)^2}{\partial^+}\psi^+ + c_6 m^2 {\psi^+}^\dagger \frac{1}{\partial^+}\psi^+ \\
&\quad + c_7 {\psi^+}^\dagger \phi \frac{1}{\partial^+}\psi^+ + c_8 {\psi^+}^\dagger \frac{1}{\partial^+}(\phi\psi^+) \\
&\quad + c_9 {\psi^+}^\dagger \phi \frac{\gamma^\perp.\partial^\perp}{\partial^+}\psi^+ + c_{10} {\psi^+}^\dagger \frac{\gamma^\perp.\partial^\perp}{\partial^+}(\phi\psi^+) \\
&\quad + c_{11} {\psi^+}^\dagger \phi \frac{1}{\partial^+}(\phi\psi^+).
\end{aligned} \tag{118}
$$

It is worthwhile to compare the above catalogue with the Hamiltonian density of the Yukawa model obtained from the Lagrangian density via the standard canonical procedure. It takes the form

$$
\begin{aligned}
\mathcal{H}_{\rm can} \; &= \frac{1}{2}\left(\partial^\perp\phi.\partial^\perp\phi + \mu^2\phi^2\right) + \lambda_3\phi^3 + \lambda_4\phi^4 \\
&\quad + {\psi^+}^\dagger \frac{\left(-(\partial^\perp)^2 + m^2\right)}{i\partial^+}\psi^+ + gm{\psi^+}^\dagger\left(\phi\frac{1}{i\partial^+}\psi^+ + \frac{1}{i\partial^+}(\phi\psi^+)\right) \\
&\quad + g{\psi^+}^\dagger\left(\phi\frac{\gamma^\perp.\partial^\perp}{\partial^+}\psi^+ - \frac{\gamma^\perp.\partial^\perp}{\partial^+}(\phi\psi^+)\right) + g^2{\psi^+}^\dagger\phi\frac{1}{i\partial^+}(\phi\psi^+).
\end{aligned} \tag{119}
$$

Comparing the forms of the Hamiltonian density constructed by two different methods, namely, the one based on light-front power counting alone and the one based on the canonical procedure starting from the Lagrangian density, it appears that the first method has too many arbitrary parameters compared to the very few parameters resulting from the second method. This should cause no surprise since the first method has relied purely on power counting whereas the second method has already implemented the consequences of Lorentz symmetries by virtue of starting from a manifestly invariant Lagrangian density. We can hope to reduce the number of free parameters by studying the implications of various symmetries in the theory. In the next section we provide an example of this idea.

### 5.4 Implementing Symmetries: A Simple Example

We have seen that the most general form of the canonical Hamiltonian density can be constructed using the power counting rules. However, the Hamiltonian density so constructed suffers from an apparent proliferation of free parameters in comparison with that obtained starting from the manifestly Lorentz invariant Lagrangian density. In this section we provide an example of how implementing symmetries implies relationship among the parameters and thus reduces the number of free parameters in the theory.

Two of the most important symmetries in light-front theory are the longitudinal and the transverse boost symmetries. As we have already observed, longitudinal boost symmetry is a scale symmetry which is already implemented in constructing the power counting rules for the canonical Hamiltonian ($P^-$ should scale as $x^-$). Transverse boost symmetry implies that interaction vertices in the theory (in momentum space) are independent of the total transverse momentum in the problem. Let us consider the consequence of this symmetry for the Hamiltonian for the Yukawa model we have constructed from power counting.

We consider the tree level matrix element for transition from a single fermion state to a fermion - boson state. Let us denote momenta of the initial fermion, final fermion and the boson by $P$, $k$, and $q$ respectively. The relevant terms of interest are those involving the transverse derivative. A simple calculation shows that, apart from common factors, the matrix element

$$\mathcal{M} \sim -c_9 \frac{\sigma^\perp . P^\perp}{P^+} - c_{10} \frac{\sigma^\perp . k^\perp}{k^+}. \tag{120}$$

Introduce the internal momenta $k^+ = xP^+$, $k^\perp = \kappa^\perp + xP^\perp$. In terms of the internal variables the matrix element

$$\mathcal{M} \sim -c_9 \frac{\sigma^\perp . P^\perp}{P^+} - c_{10} \frac{\sigma^\perp . (\kappa^\perp + xP^\perp)}{xP^+}. \tag{121}$$

Requiring that the matrix element is independent of $P^\perp$ immediately yields $c_9 = -c_{10}$. Thus the implementation of transverse boost symmetry on the transition matrix element results in the reduction of number of free parameters in the tree level Hamiltonian by one.

**Discussion.** By relying on the power counting rules rather than appealing to a manifestly Lorentz invariant Lagrangian we have a starting bare Hamiltonian that do not have the symmetries of the real world. However, demanding that the physical observables obey the symmetries we can hopefully correct our mistakes! An analysis in QED along these lines can be found in the beautiful work of French and Weisskopf (1949). An application of this idea to the problem of spontaneous symmetry breaking in sigma model on the light-front is worked out in Appendix A of Wilson et al. (1994).

The examples cited so far deals with the theory at the tree level. At this stage it looks like we are solving a simple problem in a complicated way. Fortunately, for the light-front theory matters are not so simple. As we stated in the beginning, we need to construct the most general form of the Hamiltonian i.e., the canonical terms plus the counterterms. The power counting rules we have cited are for the canonical terms. Light-front symmetries imply a far richer counterterm structure than is familiar in the equal time theory. A discussion of this structure, however, is beyond the scope of these pedagogical lectures and is the subject of active research. For a study in the context of bound state dynamics in the Yukawa model see Głazek et al. (1993). A preliminary analysis is carried out in Wilson et al. (1994). For a discussion of the reduction of free parameters in the context of light-front renormalization group see the work of Perry and Wilson (1993) and Perry (1994).

## A Notation, Conventions, and Useful Relations

We denote the four-vector $x^\mu$ by

$$x^\mu = (x^0, x^3, x^1, x^2) = (x^0, x^3, x^\perp). \tag{122}$$

Scalar product

$$x.y = x^0 y^0 - x^3 y^3 - x^\perp . y^\perp. \tag{123}$$

Define light-front variables

$$x^+ = x^0 + x^3 \ , \quad x^- = x^0 - x^3. \tag{124}$$

Let us denote the four-vector $x^\mu$ by

$$x^\mu = (x^+, x^-, x^\perp). \tag{125}$$

Scalar product

$$x.y = \frac{1}{2} x^+ y^- + \frac{1}{2} x^- y^+ - x^\perp . y^\perp. \tag{126}$$

The metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{127}$$

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{128}$$

Thus

$$x_- = \frac{1}{2}x^+, \quad x_+ = \frac{1}{2}x^-. \tag{129}$$

Partial derivatives:

$$\partial^+ = 2\partial_- = 2\frac{\partial}{\partial x^-}. \tag{130}$$

$$\partial^- = 2\partial_+ = 2\frac{\partial}{\partial x^+}. \tag{131}$$

Four-dimensional volume element:

$$d^4x = dx^0 d^2x^\perp dx^3 = \frac{1}{2}dx^+ dx^- d^2x^\perp. \tag{132}$$

Three dimensional volume element:

$$[dx] = \frac{1}{2}dx^- d^2x^\perp \tag{133}$$

Lorentz invariant volume element in momentum space:

$$[d^3k] = \frac{dk^+ \, d^2k^\perp}{2(2\pi)^3 k^+}. \tag{134}$$

The step function

$$\begin{aligned} \theta(x) &= 0, \quad x < 0 \\ &= 1, \quad x > 0. \end{aligned} \tag{135}$$

The antisymmetric step function

$$\epsilon(x) = \theta(x) - \theta(-x). \tag{136}$$

$$\frac{\partial \epsilon}{\partial x} = 2 \, \delta(x) \tag{137}$$

where $\delta(x)$ is the Dirac delta function.

$$\mid x \mid \; = \; x \, \epsilon(x). \tag{138}$$

We define the integral operators

$$\frac{1}{\partial^+} f(x^-) = \frac{1}{4} \int dy^- \epsilon(x^- - y^-) \, f(y^-), \tag{139}$$

$$(\frac{1}{\partial^+})^2 f(x^-) = \frac{1}{8} \int dy^- \mid x^- - y^- \mid \, f(y^-). \tag{140}$$

Unless otherwise specified, we choose the Bjorken and Drell convention for gamma matrices:

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{141}$$

$$\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{142}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{143}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{144}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{145}$$

$$\gamma^5 \ = \ i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{146}$$

$$\boldsymbol{\alpha} \ = \ \gamma^0\boldsymbol{\gamma}. \tag{147}$$

$$\gamma^\pm \ = \gamma^0 \pm \gamma^3. \tag{148}$$

Explicitly,

$$\gamma^\pm = \begin{pmatrix} 1 & 0 & \pm 1 & 0 \\ 0 & 1 & 0 & \mp 1 \\ \mp 1 & 0 & -1 & 0 \\ 0 & \pm 1 & 0 & -1 \end{pmatrix}. \tag{149}$$

$$\Lambda^\pm \ = \ \frac{1}{4}\gamma^\mp\gamma^\pm = \frac{1}{2}\gamma^0\gamma^\pm \ = \ \frac{1}{2}(I \pm \alpha^3). \tag{150}$$

Explicitly,

$$\Lambda^+ \ = \ \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}. \tag{151}$$

$$\Lambda^- \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{152}$$

$$(\Lambda^\pm)^2 \;=\; \Lambda^\pm. \tag{153}$$

$$(\Lambda^\pm)^\dagger \;=\; \Lambda^\pm. \tag{154}$$

$$\Lambda^+ \;+\; \Lambda^- \;=\; I. \tag{155}$$

$$\gamma^\perp\, \Lambda^\pm \;=\; \Lambda^\pm \gamma^\perp. \tag{156}$$

$$\gamma^0\, \Lambda^\pm \;=\; \Lambda^\mp \gamma^0. \tag{157}$$

$$\alpha^\perp\, \Lambda^\pm \;=\; \Lambda^\mp \alpha^\perp. \tag{158}$$

$$\gamma^5\, \Lambda^\pm \;=\; \Lambda^\pm \gamma^5. \tag{159}$$

$$\gamma^\mp = 2\Lambda^\pm \gamma^0 = \gamma^\mp \Lambda^\mp. \tag{160}$$

$$\gamma^i \Lambda^\mp = \frac{1}{2}\gamma^i \pm i\frac{1}{2}\epsilon^{ij}\gamma^j\gamma^5. \tag{161}$$

$$\alpha^j \gamma^i \Lambda^+ = \frac{i}{2}\epsilon^{ij}\gamma^+\gamma^5. \tag{162}$$

**Dirac spinors**

$$u_\lambda(k) = \sqrt{\frac{1}{m_F\, k^+}}\left[m_F\, \Lambda^- \;+\; (k^+ \;+\; \alpha^\perp.k^\perp)\, \Lambda^+\right]\, \chi_\lambda. \tag{163}$$

$$\chi_\uparrow = \sqrt{2\, m_F}\, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{164}$$

$$\chi_\downarrow = \sqrt{2\, m_F}\, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{165}$$

$$u_{\uparrow}(k) \;=\; \frac{1}{\sqrt{2k^+}} \begin{pmatrix} k^+ + m_F \\ k^1 + ik^2 \\ k^+ - m_F \\ k^1 + ik^2 \end{pmatrix}. \tag{166}$$

$$u_{\downarrow}(k) \;=\; \frac{1}{\sqrt{2k^+}} \begin{pmatrix} -k^1 + ik^2 \\ k^+ + m_F \\ k^1 - ik^2 \\ -k^+ + m_F \end{pmatrix}. \tag{167}$$

$$u^+_{\uparrow}(k^+) = \sqrt{\frac{k^+}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{168}$$

$$u^+_{\downarrow}(k^+) = \sqrt{\frac{k^+}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{169}$$

$$v_\lambda(k) = C\ (\bar{u}_\lambda(k))^T \tag{170}$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation operator.

$$v_\lambda(k) = \sqrt{\frac{1}{m_F\, k^+}} \left[ m_F\, \Lambda^- \;\;+\;\; (k^+ \;+\; \alpha^\perp.k^\perp)\ \Lambda^+ \right] \eta_\lambda. \tag{171}$$

$$\eta_{\uparrow} = \sqrt{2\, m_F} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{172}$$

$$\eta_{\downarrow} = \sqrt{2\, m_F} \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{173}$$

$$v_{\uparrow}(k) \;=\; \frac{1}{\sqrt{2k^+}} \begin{pmatrix} k^1 - ik^2 \\ -k^+ + m_F \\ -k^1 + ik^2 \\ k^+ + m_F \end{pmatrix}. \tag{174}$$

$$v_{\downarrow}(k) \;=\; \frac{1}{\sqrt{2k^+}} \begin{pmatrix} -k^+ + m_F \\ -k^1 - ik^2 \\ -k^+ - m_F \\ -k^1 - ik^2 \end{pmatrix}. \tag{175}$$

$$v^+_{\uparrow}(k) \;=\; \sqrt{\frac{k^+}{2}} \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \tag{176}$$

$$v^+_{\downarrow}(k) \;=\; \sqrt{\frac{k^+}{2}} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{177}$$

# B Survey of Light-Front Related Reviews, Books

## B.1 Review Articles on Light-Front

Several review articles have appeared touching upon various aspects of light-front dynamics. An almost complete list (till the end of 1995) follows.

The article by Rohrlich (1971) discusses quantization on the light-front together with a careful examination of the associated boundary value problem. Topics covered also include scale invariance and conformal invariance. A nice introduction to the initial value problem on the light-front is also given by Domokos (1971). Susskind (1969) and Kogut and Susskind (1973) provide the rationale for considering field theories in infinite momentum frame (IMF) with particular emphasis on high energy processes. They also discuss the nonrelativistic analogy, *i.e,* the correspondence between IMF physics and two-dimensional Galilean mechanics. Jackiw (1972) compares and contrasts the derivation of sum rules in deep inelastic scattering using a) equal time quantization together with infinite momentum techniques and b) light-cone quantization. Melosh transformation and its connection with the more familiar Pryce-Tani-Foldy-Wouthuysen transformation are reviewed by Bell (1974). Bell and Ruegg (1975) discusses the relation between relativistic parton model, non-relativistic quark model, and various SU(6) and $SU(6)_W$ broken symmetry schemes. Relativistic Hamiltonian quantum theories of finitely many degrees of freedom are reviewed by Leutwyler and Stern (1978). Phenomenological use of light-cone wavefunctions can be found in the review articles of Frankfurt and Strikman (1981) and Frankfurt and Strikman (1988). Light-cone perturbation theory and its application to various fields are reviewed by Namyslowski (1985). For applications to perturbative QCD see the review articles of Lepage, Brodsky, Huang, and Mackenzie (1983), Brodsky and Lepage (1989) and Ji (1989). An approach to hadron spectroscopy and form factors utilizing a null plane approximation to Bethe-Salpeter equation is reviewed in Chakrabarty, Gupta, Singh and Mitra (1989). Null plane dynamics of particles and fields is reviewed in Coester (1991) and Keister and Polyzou (1991). Two review articles on null plane dynamics with emphasis on covariance are Karmanov (1988) and Fuda (1991). The discretized light-cone quantization program of Brodsky and Pauli and collaborators is reviewed in Brodsky and Pauli (1991) and Brodsky, McCartor, Pauli, and Pinsky (1992). Brodsky, McCartor, Pauli, and Pinsky (1992) also has an account of the so-called zero-mode problem. An overview of the whole subject is given by Ji (1992). Reviews of light-front dynamics with emphasis on renormalization problem are given by Głazek (1993) and by Perry (1994). A detailed review with emphasis on QCD and phenomenology of hadron structure is given by Zhang (1994). For review of light-front dynamics with detailed discussion of the aspects of zero mode problem, see, Burkardt (1995).

### B.2 Light-Front in Books

Light-front dynamics has made its entry into a few books. In the following, we have omitted standard textbooks that introduce light-front variables in the context of deep inelastic scattering.

A very brief treatment appears in *The Theory of Photons and Electrons: The Relativistic Quantum Field Theory of Charged Particles with Spin One-Half*, Expanded Second Edition, J.M. Jauch and F. Rohrlich, (Springer-Verlag, New York, 1976).

In the context of current algebra and deep inelastic scattering, light-front dynamics appears in *Currents in Hadron Physics*, V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, (North-Holland Publishing Company, Amsterdam, 1973). This book also provides an excellent discussion of the infinite-momentum limit. Also, see, *Theory of Lepton-Hadron Processes at High Energies: Partons, Scale Invariance and Light-Cone Physics*, P. Roy, (Clarendon Press, Oxford, 1975).

Speaking of deep inelastic scattering, one should not forget partons. The classic reference is *Photon-Hadron Interactions*, R.P. Feynman, (Benjamin, Reading, MA 1972).

For the utility of light-front variables in high energy scattering in the context of high orders of Feynman diagrams, see, *Expanding Protons: Scattering at High Energies*, H. Cheng and T.T. Wu, (The M.I.T. Press, Cambridge, Massachusetts, 1987).

In the context of Poincare Group and relativistic harmonic oscillator, see, *Theory and Applications of the Poincare Group*, Y.S. Kim and M.E. Noz, (D. Reidel Publishing, Dordrecht, Holland, 1988).

For the application of light-front formalism to relativistic nuclear physics, see, *Relativistic Nuclear Physics in the Light-Front Formalism*, V.R. Garsevanishvili and Z.R. Menteshashvili, (Nova Science Publishers Inc., New York, 11725, 1993).

The following workshop proceedings deal with light-front dynamics.

1. *Nuclear and Particle Physics on the Light Cone*, edited by M.B. Johnson and L.S. Kisslinger, (World Scientific, Singapore, 1989).
2. *Theory of Hadrons and Light-Front QCD*, edited by St. D. Głazek, (World Scientific, Singapore, 1995).

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