Statistical Mechanics : Problem set-II

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1. **Corrections :**

$$C_\xi(u) = \langle e^{iu\xi}\rangle = 1 + \sum_{n=1}^{\infty} \frac{(iu)^n}{n!} M_n = \exp\left(\sum_{n=1}^{\infty} \frac{(iu)^n}{n!} K_n\right)$$

where M_n are the moments. The above equation, in fact, define the cumulates K_n. Now let us define $\chi_n = \langle(\xi - M_1)^n\rangle$. We have shown in the class, $K_1 = M_1, K_2 = \chi_2$. Show that, $K_3 = \chi_3$ and $K_4 = \chi_4 - 3\chi_2^2$

2. **Simple Distribution :** Find M_n, χ_n and K_n of a square distribution defined by
$P(x) = 0$ for $|x| > a$ and $P(x) = \frac{1}{2a}$ for $|x| < a$.

3. **First order phase transition :** In a first order phase transition, two phases (the ordered and the dis-ordered) co-exist at the transition point. The distribution of energy may be written as,

$$P(E) = c\delta(E - E_o) + (1 - c)\delta(E - E_d),$$

where E_o and E_d are energy of the ordered and the disordered phase respectively and $0 \le c \le 1$ is a measure of order. Note that the system is completely ordered (disordered) when $c = 1$ ($c = 0$). Show that near transition (when c or $(1 - c)$ is small), the Fourth order cumulant is positive. (Hint : show that $\frac{K_4}{3\chi_2^2} = \frac{1}{3c(1-c)} - 2)$).)

4. **Harmonic oscillator :** A harmonic oscillator with levels $h\nu$ $(n = 0, 1, 2 \ldots)$ has in thermal equilibrium the probability

$$P_n = (1 - \gamma)\gamma^n, \qquad (1)$$

to be in level n, where $\gamma = e^{-h\nu/k_BT}$. Find the moments and cumulants of this distribution.

5. **Generating Function (again):** The generating function for any given distribution $P(x)$ is defined by $F(z) = \langle z^x\rangle$.
(a) What is $F(z)$ for Eq. (1) ?
(b) Can you find out $P(x)$ is $F(z)$ is given ?
(c) What is $P(x)$ for $F(z) = \frac{1}{1-z}$

6. **Fractal Dimension :** Start with a n-polygon with side a. Divide each line segment into three equal parts, remove the middle and **raise** a new n-polygon with side same as the removed segment. Repeat this process to generate a fractal. What is the fractal dimension d of this object ? Show that area bounded by this object diverges when $d = 2$.

7. **Stochastic fractals :** Let's take the example of square (4-polygon). Instead of **raising** a square in each step, now sometimes you draw it out wards (with probability p) and sometimes in wards (prob $1-p$). You would generate a figure whose area A (after ∞ steps) is a random variable. What is the average area ? Can you guess (you do not have to calculate) what is the distribution $P(A)$?

8. **Scale Invariance :** A mathematical statement of scale invariance is $f(\lambda x)$ = constant $f(x)$. (Thus, fractals are scale invariant). Show that $f(x)$ necessarily has the form $f(x) \sim x^{\alpha}$.

9. **Uniform distribution:** Let ξ be a random number distributed uniformly in $[0, 1]$.
(a) Draw n random numbers from a uniform distribution in $[0, 1]$. What is the probability that n numbers drawn from this distribution are all larger than y ?
(b) How is $\eta = \xi_1 + \xi_2$ distributed ?
(c) What is the distribution of $\chi = \xi_1 \xi_2$?
(d) What function of ξ is distributed exponentially ?

10. **Independent variables ? :** x and y are random variables defined in $(0, 1)$. The joint probability distribution is $P(x, y) = \mathcal{N}(y^2 - x^2)$ when $y < x$ or otherwise $P(x, y) = 0$. Here N is the normalization constant (calculate it). What are $P(x)$, $P(y)$ and $P(x|y)$ and $P(y|x)$. Are x and y independent variables ? If not, what is the correlation between them ?

11. **Probability transformation:** Let η_1 and η_2 are the random variables drawn from the same distribution, say $P_{\eta}(x)$. Show that two new random variables $\xi = a\eta_1 + b\eta_2$ and $\chi = c\eta_1 + d\eta_2$ are independent if the vectors (a, b) and (c, d) are orthogonal.