Simple models for the distribution of wealth

Marco Patriarca

Laboratory of Computational Engineering
Helsinki University of Technology, Finland
www.lce.hut.fi

Collaboration and sponsor:

Anirban Chakraborti [1],

Kimmo Kaski [2]

Guido Germano [3]

Marco Patriarca [2,3]

1. *Department of Physics, Brookhaven National Laboratory,*
Upton, New York 11973, USA

2. *Laboratory of Computational Engineering, Helsinki University of Technology,*
P.O. Box 9203, 02015 HUT, Finland

3. *Physical Chemistry, Philipps University Marburg,*
35032 Marburg, Germany

Sponsor: *Academy of Finland*

Motivations

- Is it possible to construct a simple quantitative microscopic model of a market economy which can reproduce some relevant features such as money (wealth) distribution?

- Money: conserved, quantitatively well defined, a possible measure of economic activity

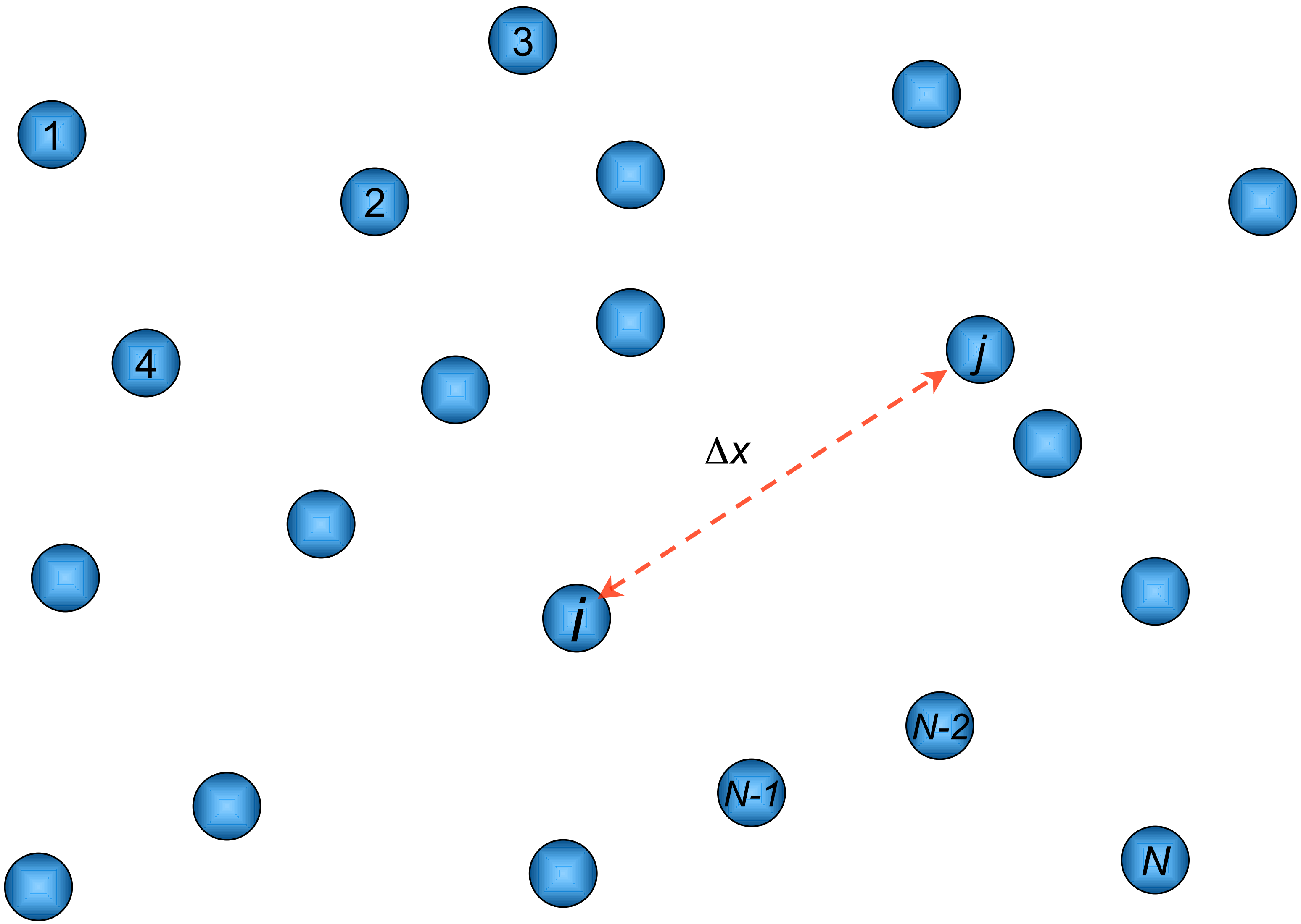
Topics

1. Generalities about the kinetic (gas-like) model: the money redistribution is assimilated to the dynamics of a perfect gas, and trade to energy exchange during collisions.

2. Basic model: the average money $\langle x \rangle$ is directly related to temperature T.

3. Model with global saving propensity $\lambda \in (0,1)$: λ defines an effective dimension $D(\lambda)$.

4. Model with individual saving propensities $\lambda_i \in (0,1)$: qualitatively new phenomena take place which modify the wealth distribution.



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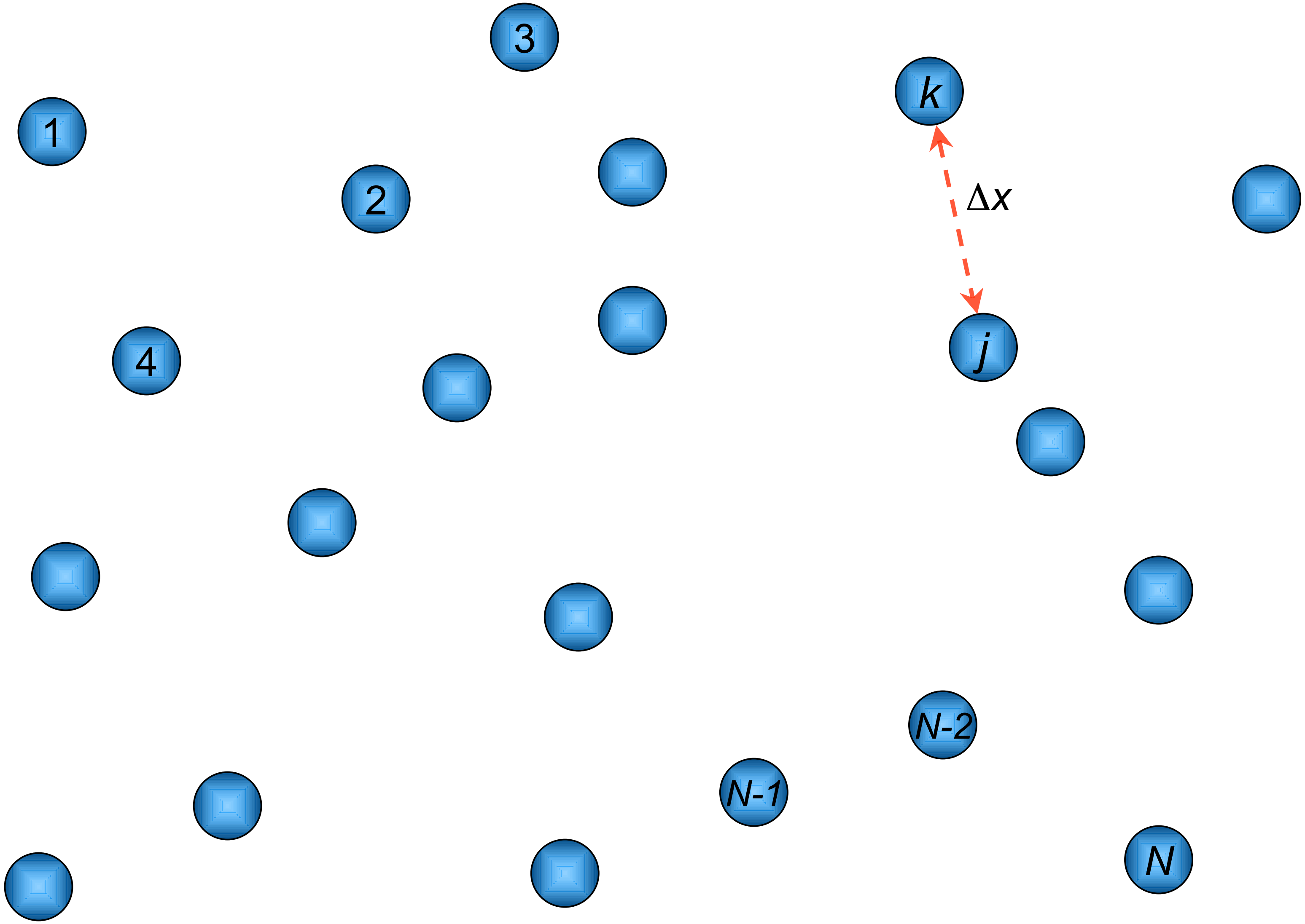
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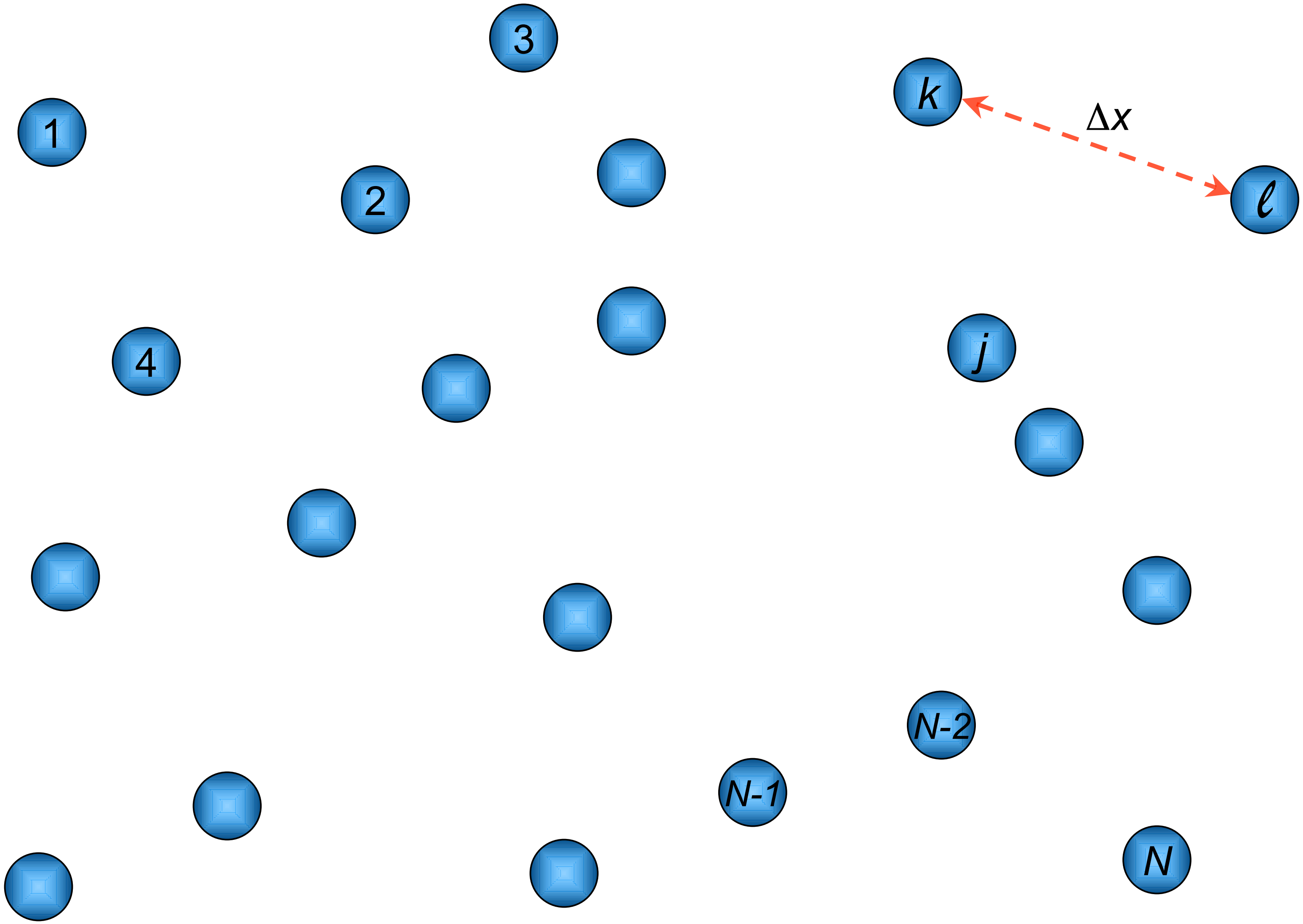
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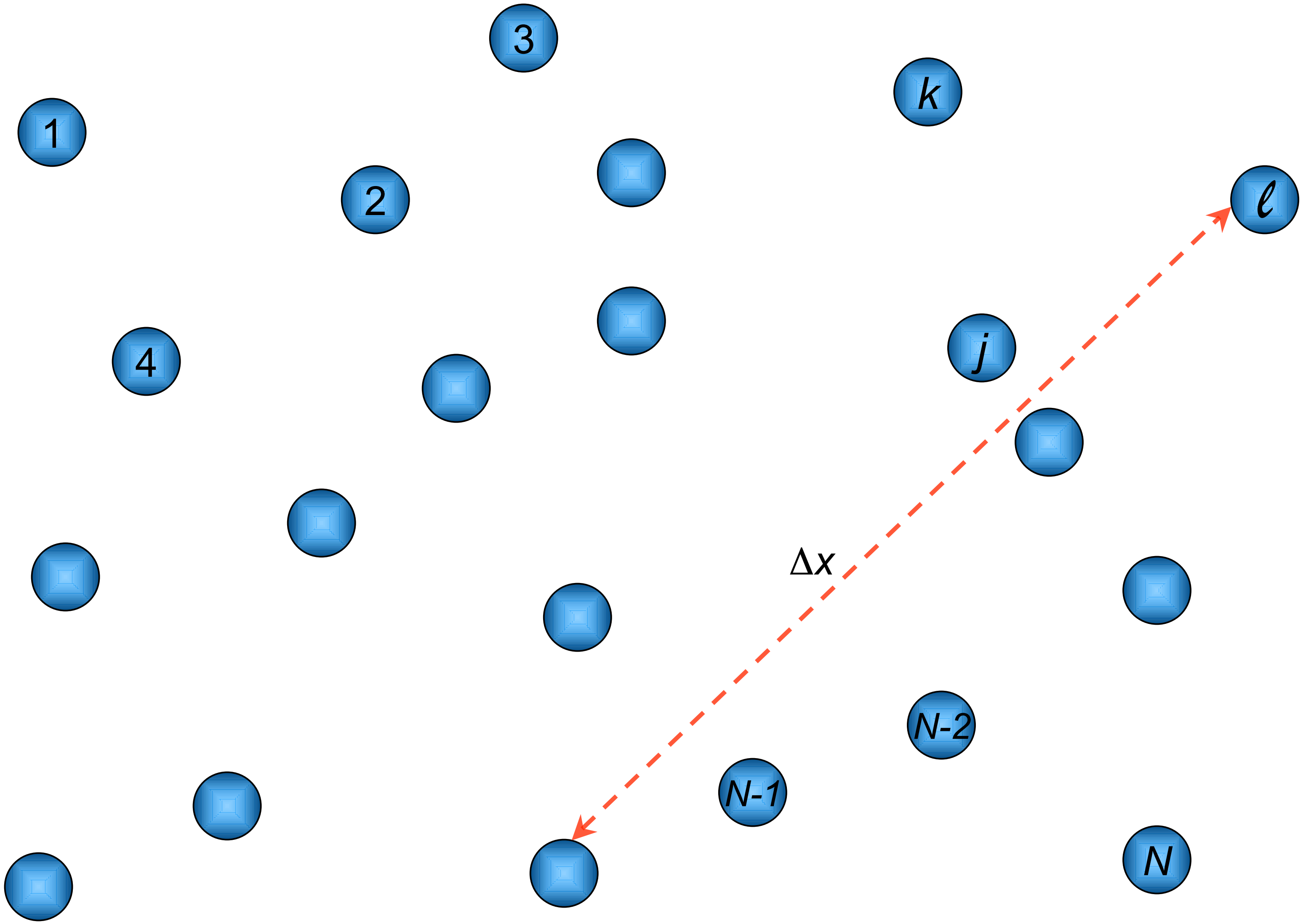
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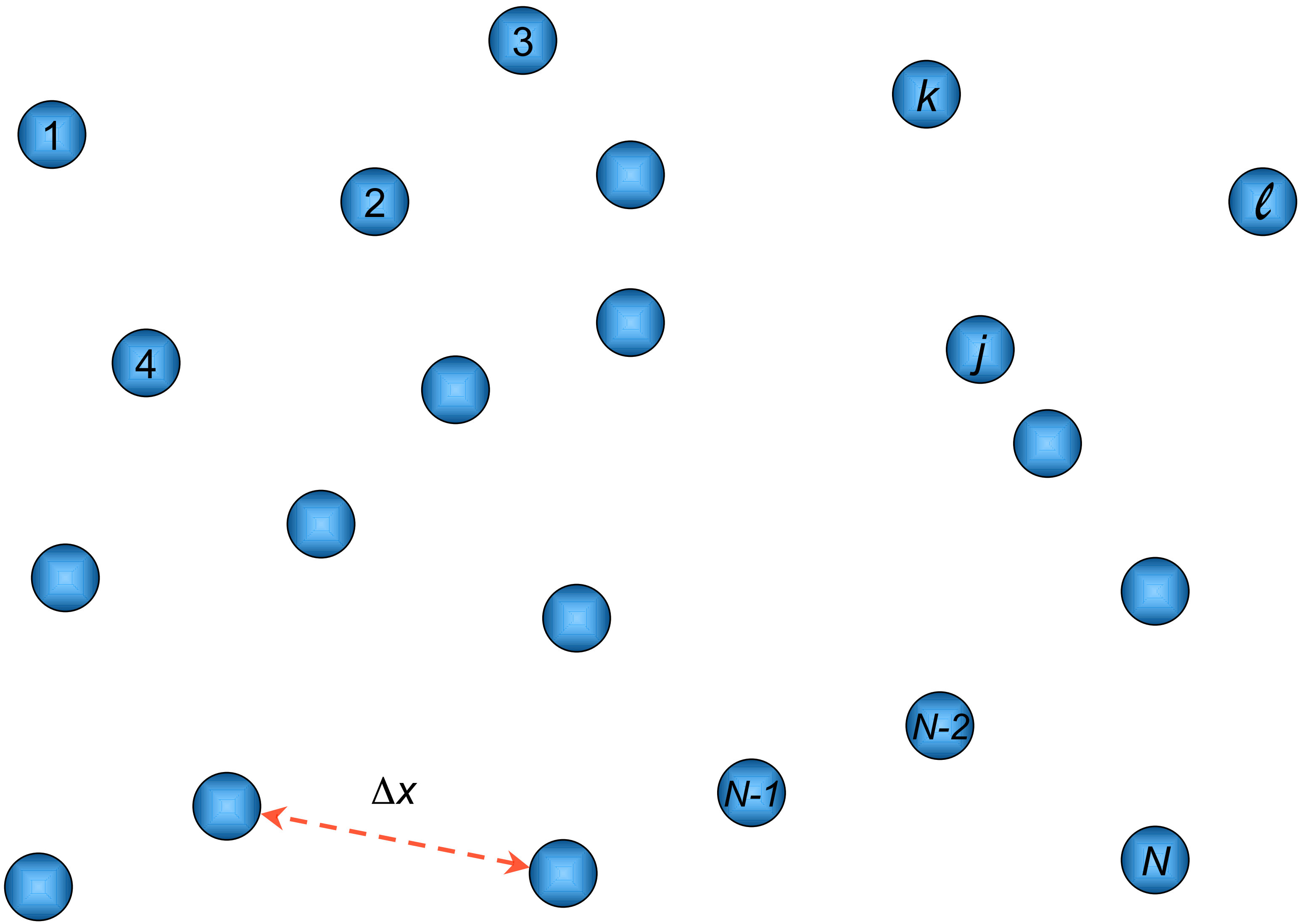

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I. Basic Model System [1]

- N units (agents)
- Assign initial wealth $\{x_i\}$
- At every time step t two agents k and j are extracted at random
- x is re-distributed at random between k and j according to a dynamical money-conserving (stochastic) dynamics

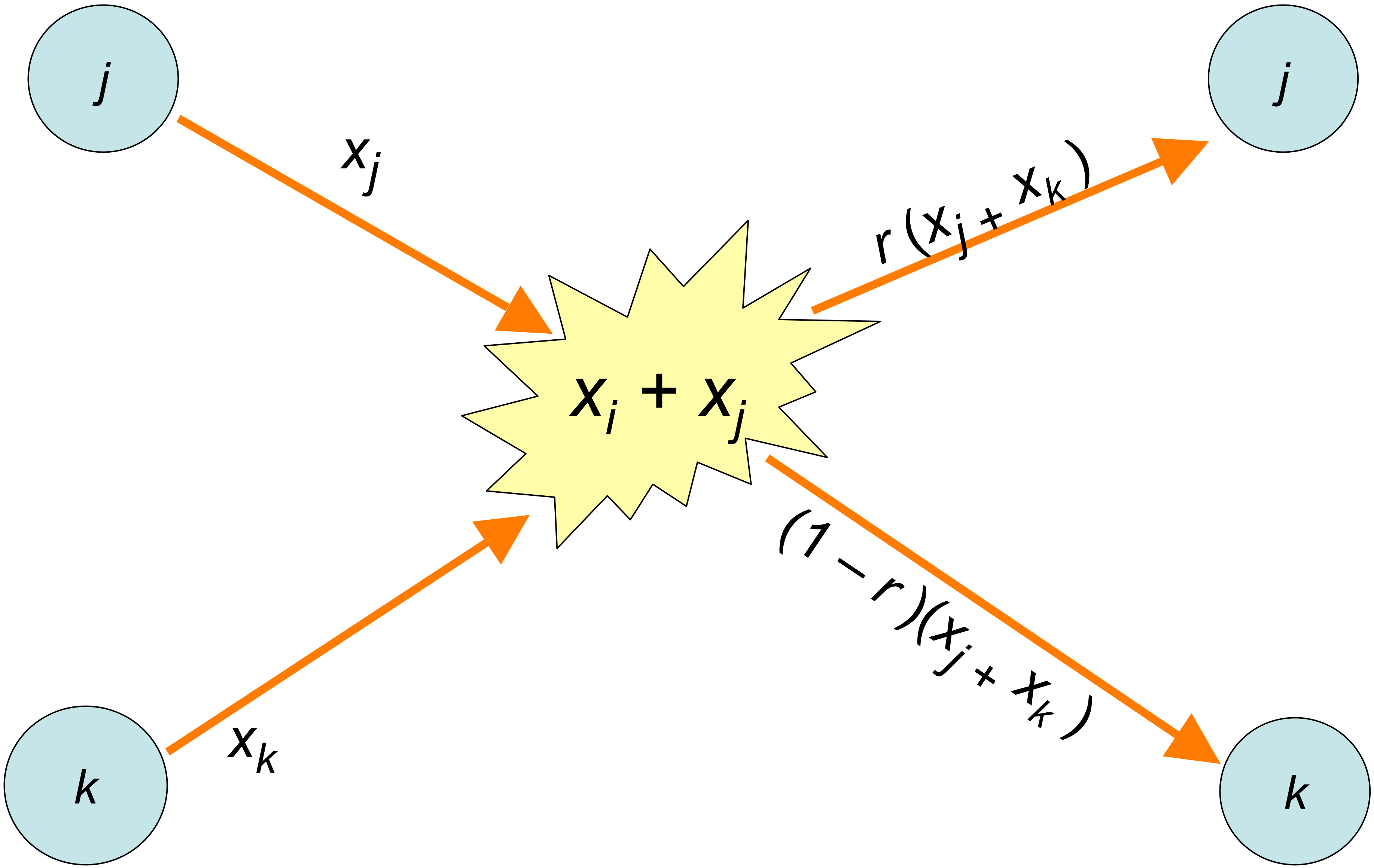
  (r is a random number between 0 and 1)

$$
\begin{aligned}
x_k &\rightarrow x_k + r(x_k + x_i) \\
x_j &\rightarrow x_j + (1-r)(x_k + x_i)
\end{aligned}
$$

- Time evolution is carried out until thermal equilibrium is reached

[1] A. Dragulescu and V. M. Yakovenko, *Statistical mechanics of money*, Eur. Phys. J. B 17 (2000) 723.

Analogy between wealth exchange between agents and energy exchange between particles (r = random number in [0,1])



Equilibrium Distribution:

$$f(x) = \frac{1}{\langle x \rangle} \exp\left(-\frac{x}{\langle x \rangle}\right)$$

i.e. the Boltzmann distribution, where $\langle x \rangle$ is the average value of x.

[1] Random Δx:
• A. Dragulescu and V. M. Yakovenko, *Statistical mechanics of money*, Eur. Phys. J. B 17 (2000) 723.

[2] Constant Δx:
• E. Bennati, *La simulazione statistica nell'analisi della distribuzione del reddito: modelli realistici e metodo di Montecarlo*, ETS Editrice, Pisa,1988.
• E. Bennati, *Un metodo di simulazione statistica nell'analisi della distribuzione del reddito*, Rivista Internazionale di Scienze Economiche e commerciali, August (1988) 735
• E. Bennati, *Il metodo Montecarlo nell'analisi economica,* Rassegna di lavori dell'ISCO (4) (1993) 31

The model is robust in that the corresponding Gibbs distribution is obtained in different conditions:

- Different initial distributions of x
- Pairwise as well as multi-agent interactions
- Constant as well as random Δx
- Random, first neighbor, as well as consecutive selection of the interacting agents
- Various linear forms of Δx
- Rapid convergence to equilibrium distribution also for a very small number of agents

II. Model with saving propensity λ [1,2,3]

$(0 < \lambda < 1)$

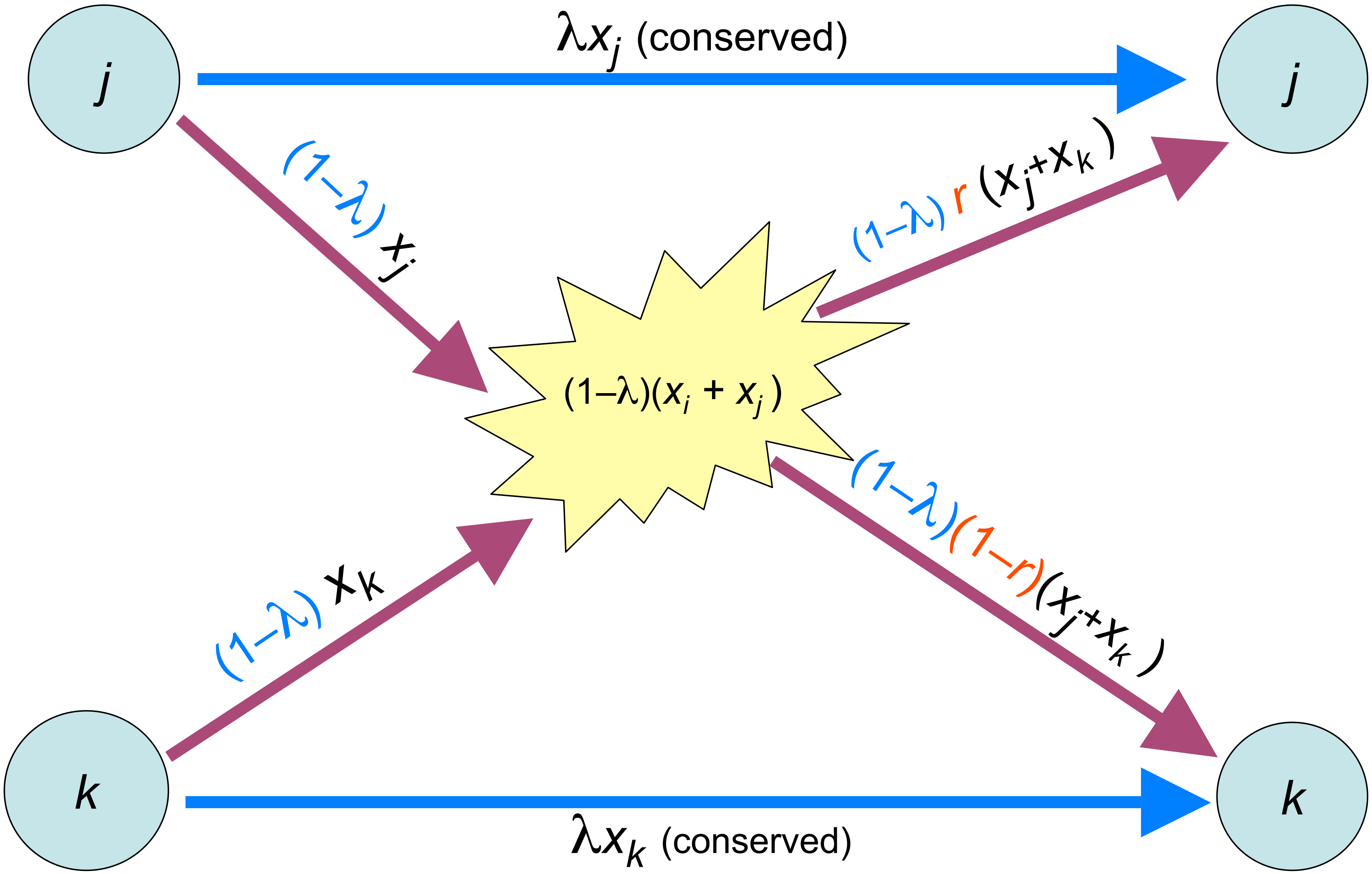
- N units (agents) with a wealths $\{x_i\}$
- At every time step t extract randomly two agents k and j .
- x is then redistributed randomly between k and j according to a dynamical money-conserving (stochastic) dynamics

$$\begin{aligned} x_k &\rightarrow \lambda x_k + (1-\lambda) r (x_k + x_i) \\ x_j &\rightarrow \lambda x_j + (1-\lambda)(1-r)(x_k + x_i) \end{aligned}$$

- Time evolution is carried out until thermal equilibrium is reached.
- x is still conserved, but only a money fraction ($1 - \lambda$) is exchanged in a single trade.

- [1] A. Chakraborti, PhD Thesis.
- [2] A. Chakraborti and B. K. Chakrabarti, Eur. Phys. J. B **17**, 167 (2000).
- [3] A. Chakraborti, Int. J. Mod. Phys. C **13**, 1315 (2002).

Visualization of x exchange: case $\lambda > 0$ (r = random number in [0,1])



Equilibrium Distribution

The equilibrium distribution is a gamma distribution

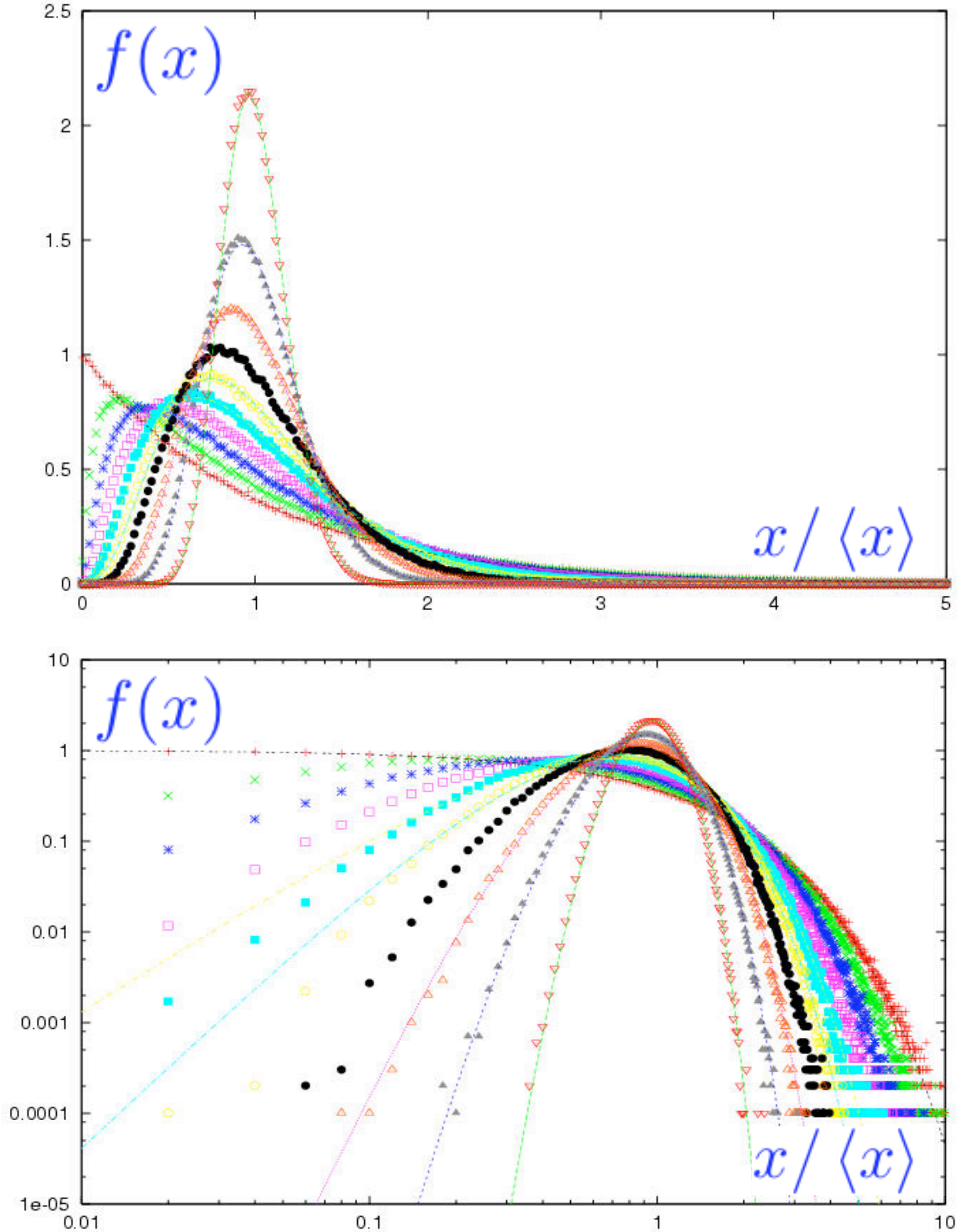
$$f(x) = a_n x^{n-1} \exp\left(-\frac{nx}{\langle x\rangle}\right)$$

where ‹*x*› is the average *x*,

$$n = 1 + \frac{3\lambda}{1-\lambda}$$

The normalization constant is

$$a_n = \frac{1}{\Gamma(n)}\left(\frac{n}{\langle x\rangle}\right)^n$$



Heuristic argument:

- Two particles colliding in an N-dimensional space will exchange only a fraction $\Delta x/x$ of the order of $1/N$ of their total kinetic energy x.

- The rest, that is the energy $(1 - 1/N)\, x$, is saved.

- We expect a similar $\lambda \approx \Delta x/x \approx 1 - 1/N$, for the "energy saving propensity" λ.

- Compare with the findings from numerical fitting, by which we find the following formula for the power n,

$$\lambda = 1 - \frac{1}{\frac{n}{3} + \frac{2}{3}} \qquad \text{or} \qquad n = 1 + \frac{3\lambda}{1 - \lambda}$$

Making the analogy more precise: Maxwell-Boltzmann distribution in *D* dimensions

Start from the single-particle Maxwell-Boltzmann distribution in *D*-dimensions:

$$f(v) = (2T/m\pi)^{D/2} \exp\left(-mv^2/2T\right)$$

$$v^2 \equiv \mathbf{v}^2 = \sum_{k=1}^{D} v_k^2$$

Integrate the angular varibles using the sphere hypersurface in *D* dimensions:

$$S_D(v) = \frac{2\pi^{D/2}}{\Gamma(D/2)}\, v^{D-1}\,, \quad \int_\Omega dv^D f(v) = dv S_D(v) f(v)\,.$$

Change variable from velocity modulus *v* to kinetic energy $x = mv^2/2$:,

$$f(x) = \frac{T}{\Gamma(D/2)} \left(\frac{x}{T}\right)^{D/2-1} \exp\left(-\frac{x}{T}\right) \equiv T\gamma_{D/2}\left(\frac{x}{T}\right)$$

This is the *gamma distribution* $\gamma_n(\xi)$ for $\xi = x/T$ with index $n = D/2$.

Compare:

wealth

$$f(x) = a_n x^{n-1} \exp\left(-\frac{nx}{\langle x \rangle}\right)$$

where $n = 1 + 3\lambda / (1 - \lambda)$,
λ is the saving propensity, and
$\langle x \rangle$ the average wealth.

energy

$$f(x) = \frac{T}{\Gamma(D/2)} \left(\frac{x}{T}\right)^{D/2-1} \exp\left(-\frac{x}{T}\right)$$

where D is the number of dimensions and
T the temperature.

Economy model	Gas model
x = money	K = kinetic energy
N-agent system	N-particle system
Trades	Interactions
$\lambda \rightarrow$ Effective dimension $D = 2\,(1 + 2\,\lambda)\,/\,(1 - \lambda)$	Space dimension D
Effective temperature $T = 2\,\langle x\rangle\,/\,D$ $\approx (1 - \lambda)\,\langle x\rangle$	Temperature $k_B T = 2\,\langle K\rangle\,/\,D$
$\xi = x\,/\,T$	$\xi = K\,/\,T$
$f(\xi) = \gamma_{D/2}(\xi) = \dfrac{1}{\Gamma(D/2)}\,\xi^{D/2-1}\mathrm{e}^{-\xi}$	

Meaning of effective temperature

Effective temperature: $T = 2 \langle x \rangle / D$

Effective dimension $D = 2\,(1 + 2\,\lambda) / (1 - \lambda)$

$T = 2 \langle x \rangle / D = (1 - \lambda) \langle x \rangle / (1 + 2\lambda) \approx (1 - \lambda) \langle x \rangle$

- Temperature is an estimate of the actual money fluctuations in a single trade

The Boltzmann equation approach [1]

A possible approach to a more rigorous demonstration of the conjecture illustrated above for the relation between the effective dimension $N = 2n$ and λ has been suggested by Repetowicz, Hutzler, and Richmond [1].

Within the framework of mean field theory they showed that the model leads to an equilibrium distribution with the first 2 moments identical to those of the gamma distribution.

- [1] P. Repetowicz, S. Hutzler, and P. Richmond, *Dynamics of Money and Income Distributions,* arXiv:cond-mat/0407770

III. Model with individual saving propensity λ_n [1,2]

$(0 < \lambda_n < 1)$

- N units (agents) with wealths $\{x_i\}$
- At every time step t extract randomly two agents k and j .
- Wealth is then redistributed randomly between k and j according to a dynamical money-conserving (stochastic) dynamics

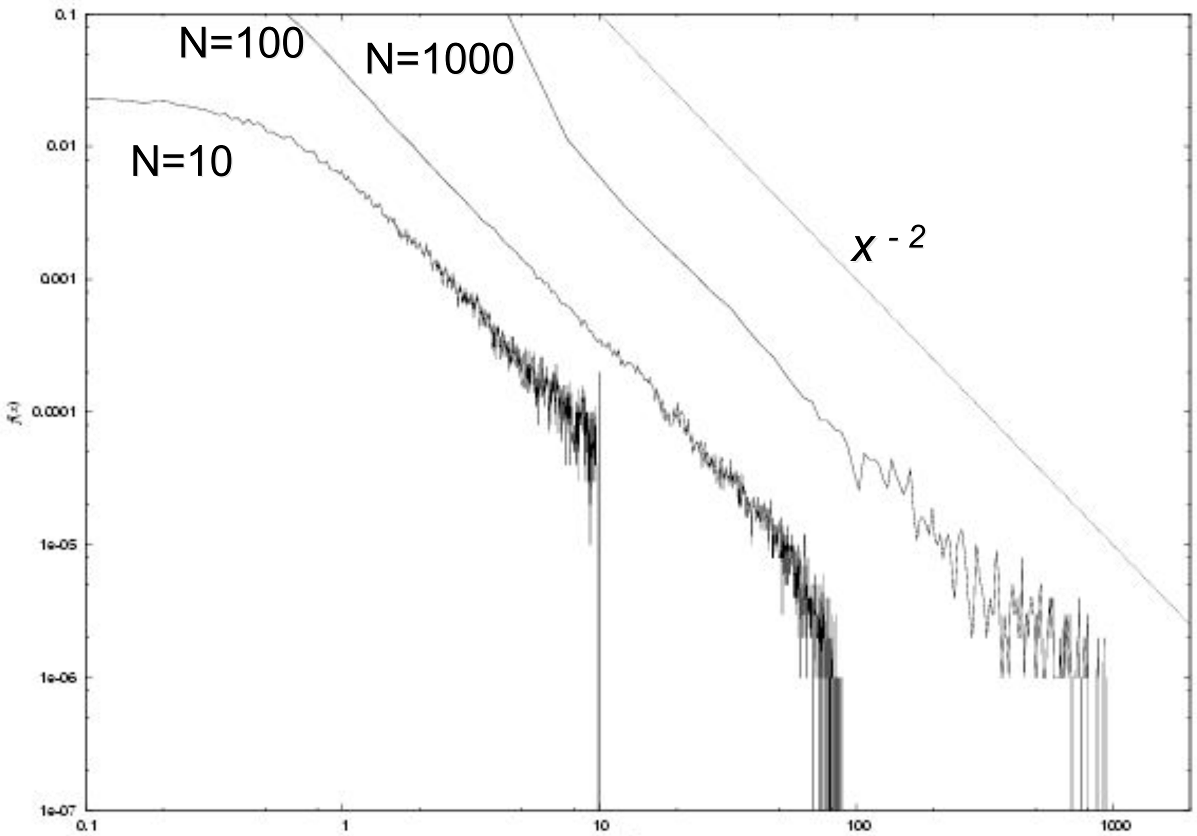
$$
\begin{aligned}
x_k &\rightarrow \lambda_k x_k + r[(1-\lambda_k)x_k + (1-\lambda_j)x_i) \\
x_j &\rightarrow \lambda_j x_j + (1-r)[(1-\lambda_k)x_k + (1-\lambda_j)x_i]
\end{aligned}
$$

- Time evolution is carried out until thermal equilibrium is reached.

- x is still conserved, but only a fraction dependent on the specific agents i and j is exchanged in a single step.

A simple recipe for a power law [1,2,3]

- Choose a random initial saving propensity distribution $\{\lambda_n\}$
- Equilibrate the system through the trading-dynamics
- Reassign randomly $\{\lambda_n\}$
- Repeat these steps and take the average over equilibrium configurations



- **Why is this procedure necessary if all agents are equivalent to each other?**
- **And why does this procedure work ?**

[1] A. Chatterjee and B. K. Chakrabarti and S. S. Manna, *Money in Gas-Like Markets: Gibbs and Pareto Laws*, Physica Scripta T 106 (2003) 367

[2] A. Chatterjee and B. K. Chakrabarti and S. S. Manna, *Pareto law in a kinetic model of market with random saving propensity*, Physica A 335 (2004) 155

[3] A. Chatterjee and B. K. Chakrabarti and R. B. Stinchcombe, *Master equation for a kinetic model of trading market and its analytic solution*,cond-mat/0501413

Why is this procedure necessary ?

Correlation between wealth and saving propensity

In models with individual saving propensity there is a correlation between the individual saving propensities λ_n and the corresponding wealth x_n .

This happens quite generally, for random as well as deterministic assignement of λ_n , for power or nonpower laws, for all distributions of λ_n .

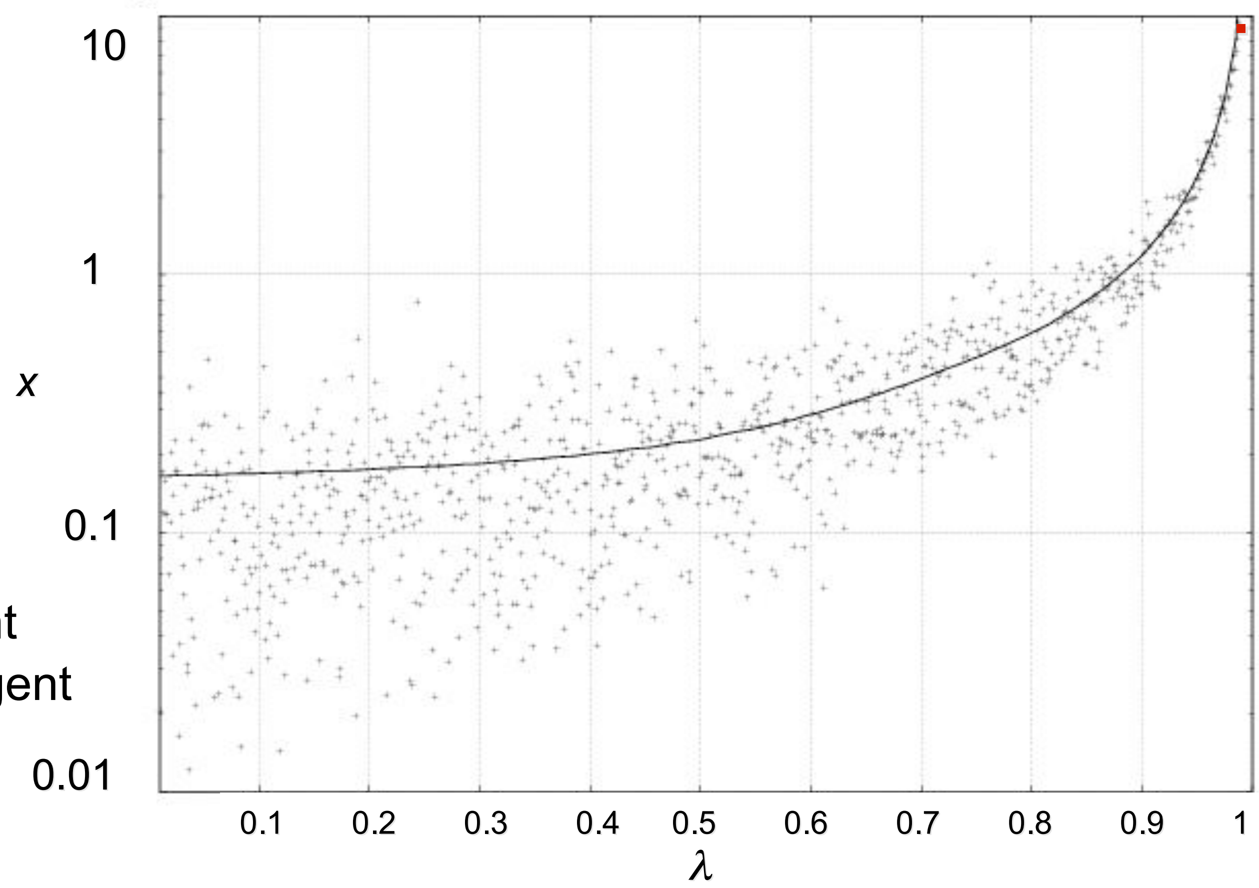
This is a typical plot

of the correlation →

- dots are single agents,

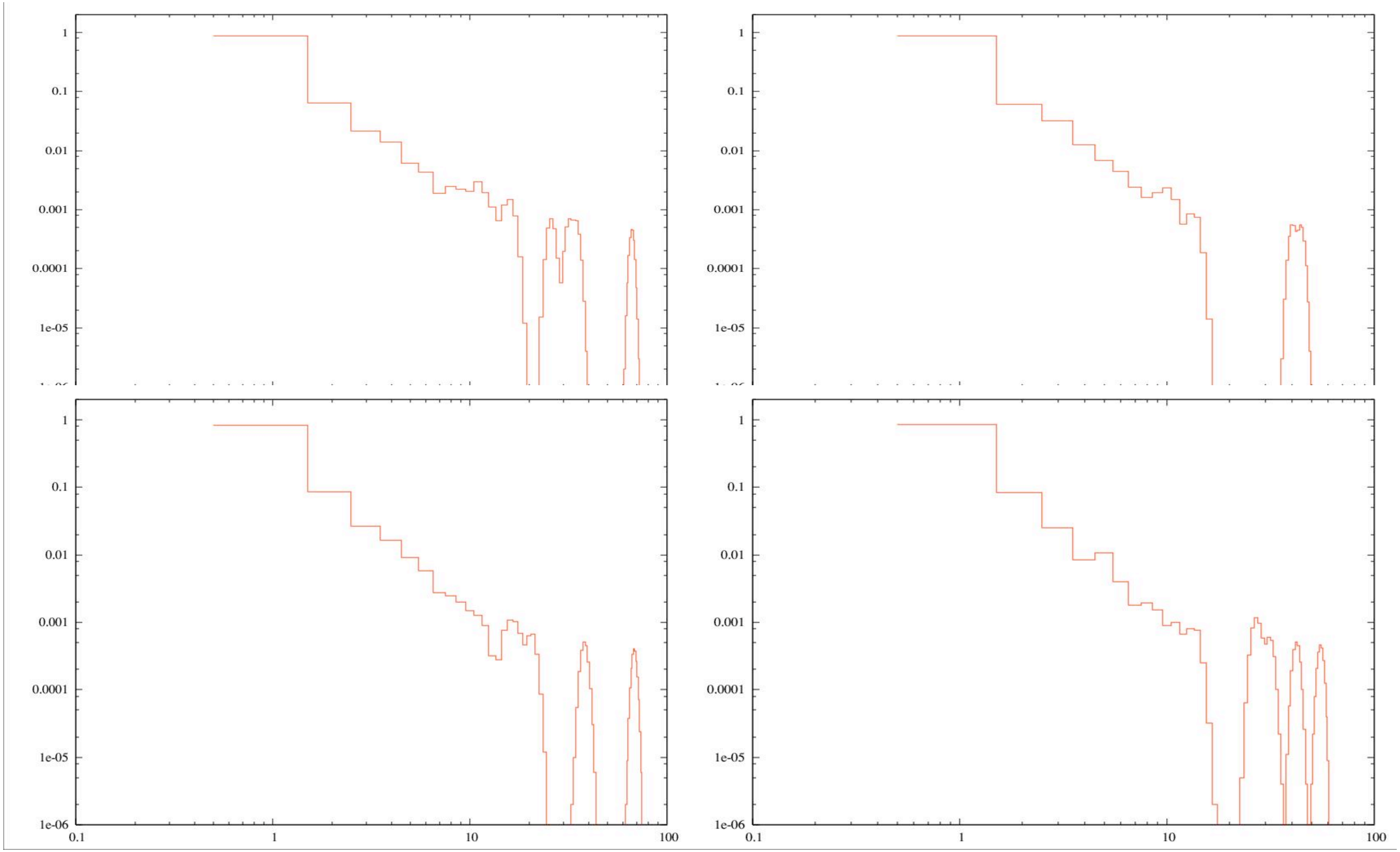
- the continuous line is the average over *x*.

- The λ_n do not represent the strategy of a single agent (no game theory) but the effective fraction of saved money

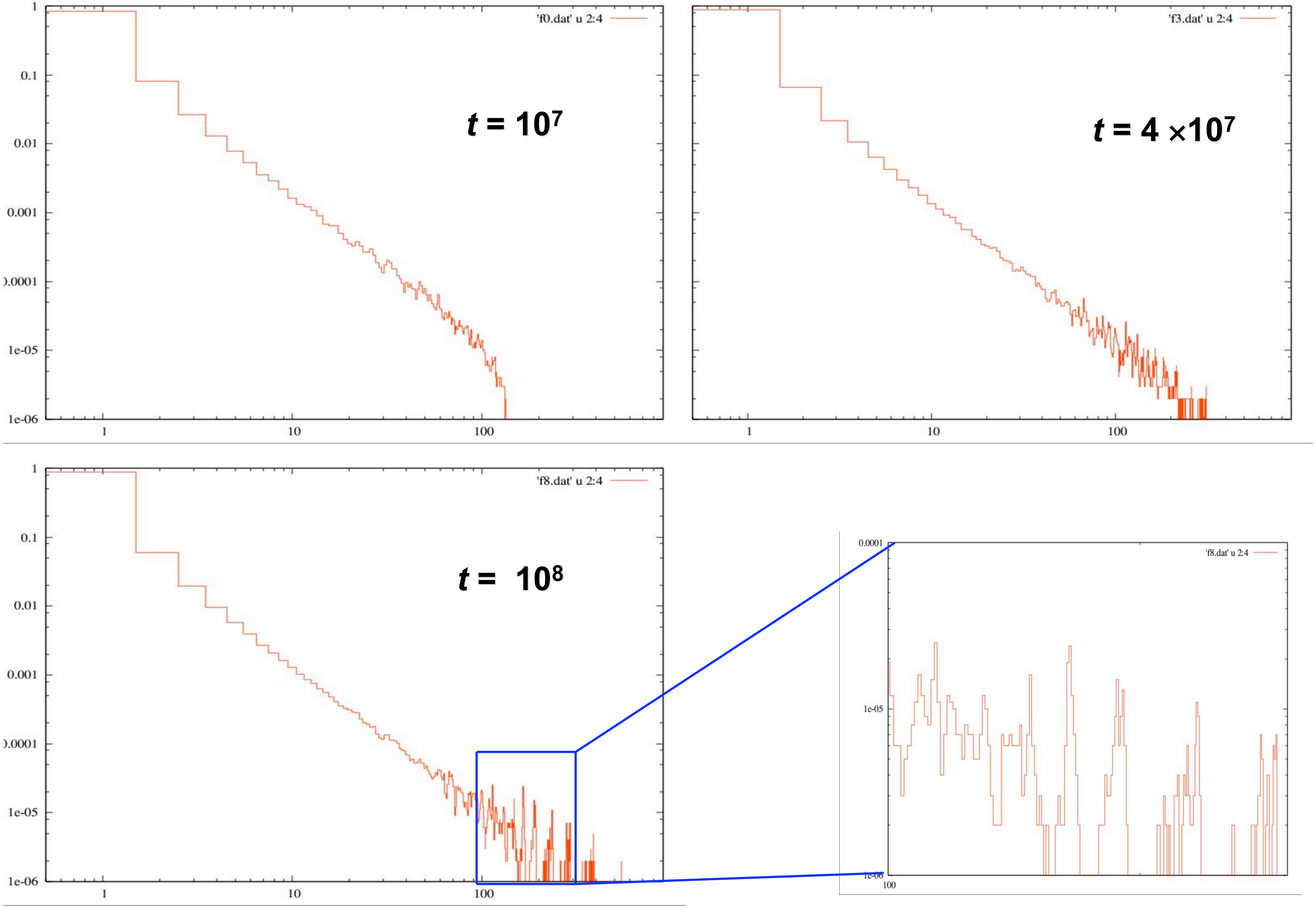


Why is the average procedure required?

Different equilibrium distributions for different sets $\{\lambda_n\}$



Time evolution for a fixed set of random saving propensities $\{ \lambda_n \}$ (500 agents)



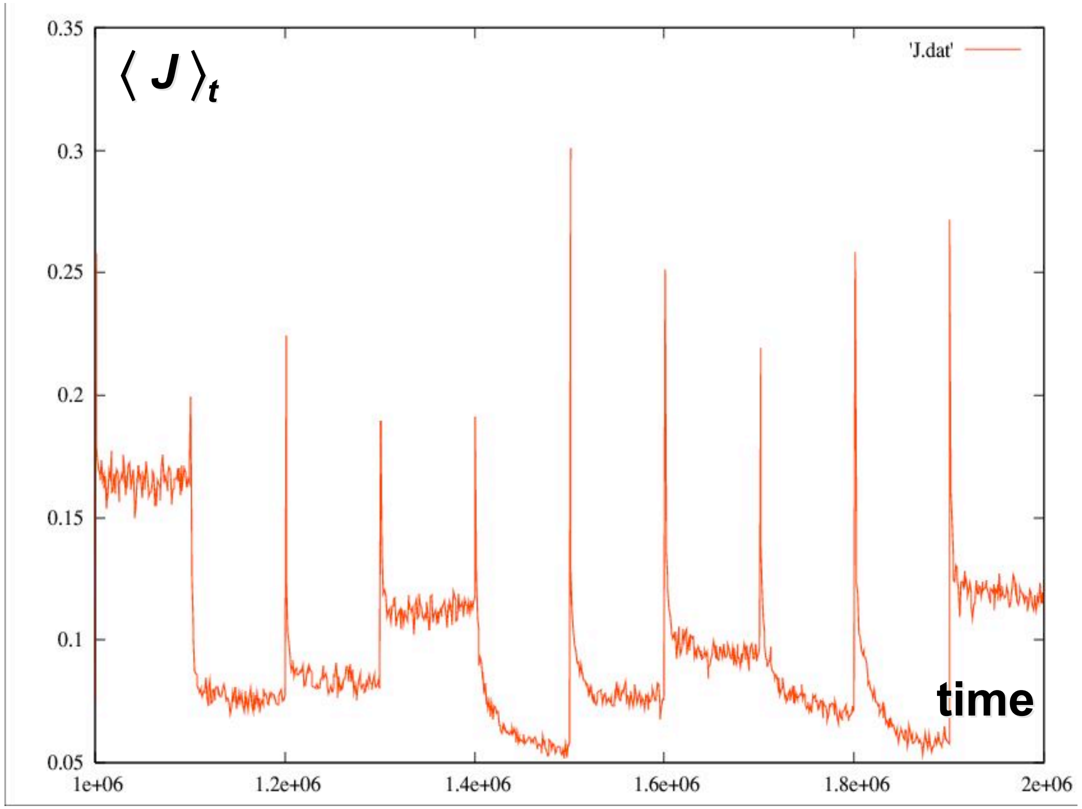
flow during time evolution

- Reassignement of the λ_n brings the system out of equilibrium
- Then the system relaxes toward the new equilibrium configuration

Define $$\langle J \rangle_t = \frac{1}{\tau} \sum_{j=1}^{\tau} |x_j(t) - x_j(t-1)|$$

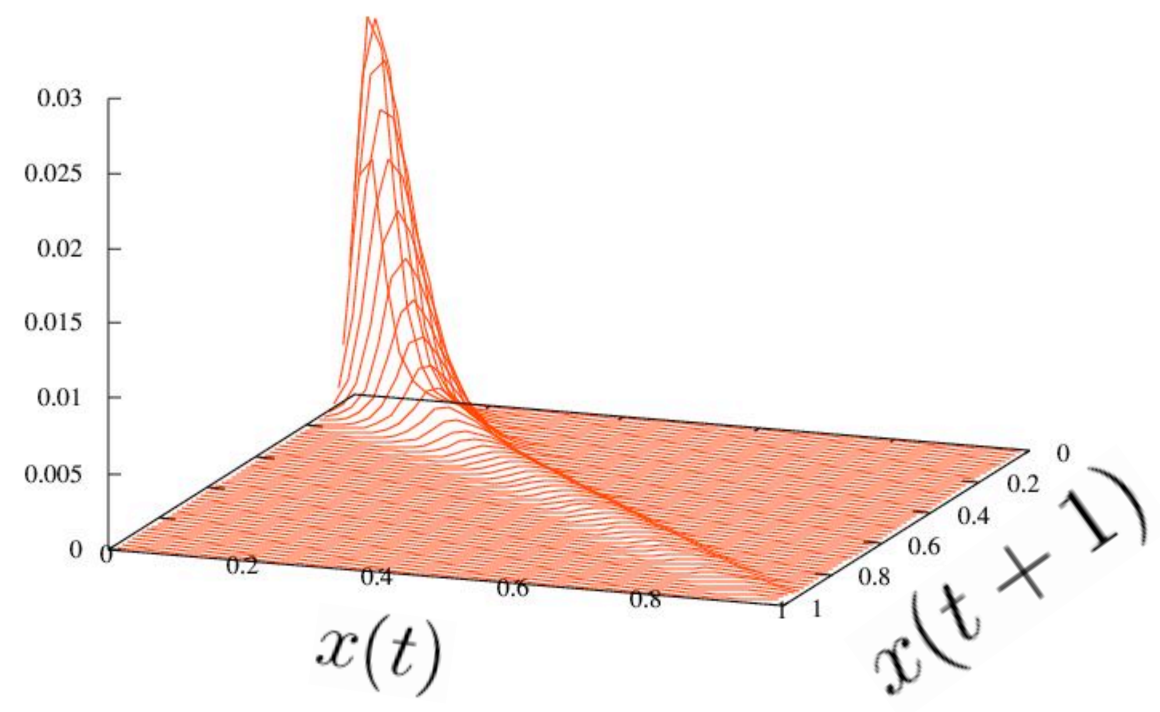
- $\langle J \rangle_t$ shows peaks in correspondence
of the reassignement of the saving propensities.

- Notice that the value of $\langle J \rangle_t$ at equilibrium is different for different configurations



Richer agents remain richer

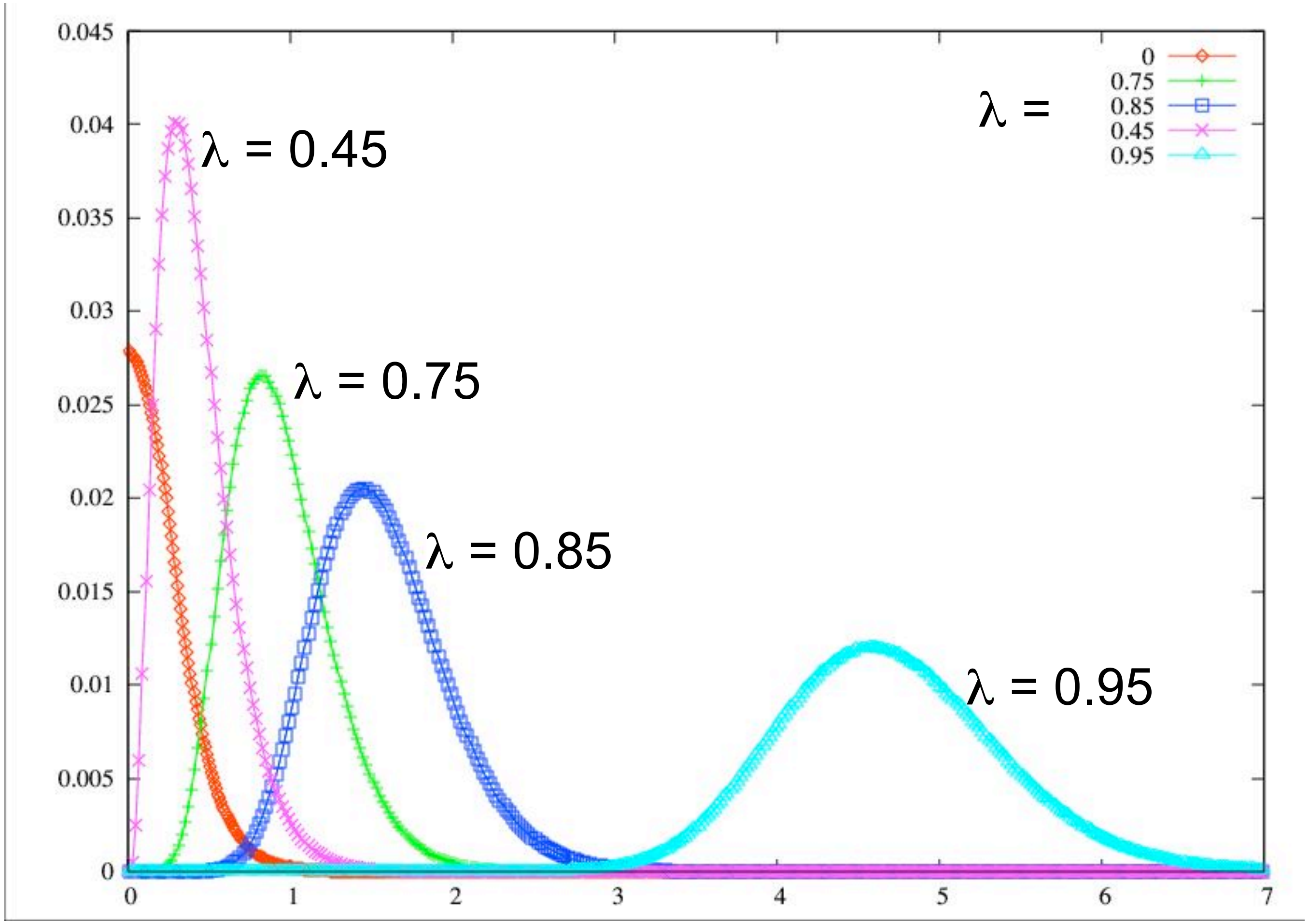
(only poor agent money vary appreciably)



- Notice that rich agents will never risk all their money: they have a large saving propensity ($\lambda \approx 1$) and therefore a very low effective temperature $T \approx (1- \lambda)$
- Thus they only invest a small amount of money in a trade
- This is shown by the small width of the $x(t+1)$-$x(t)$ map at large values of x.

Power law as superposition of exponential-distributions

- notice the shift in the mode: the subsystems with fixed λ are now open



Conclusions

- Subsystems with a given λ always at equilibrium with exponential distributions, even when arbitrary individual saving propensities λ_i are assigned

- They behave has open, coupled subsystems with their own temperature *T* and effective dimension *D*.

- Power law obtained from superpositions of such distributions

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- [ ] B. K. Chakrabarti and S. Marjit, Ind. J. Phys. B 69 (1995) 681

- Basic model:
- [3] A. Dragulescu and V. M. Yakovenko, Physica A **299**, 213 (2001).
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- Directly related to this talk:
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- [21] *Dynamics of Money and Income Distributions***,** P. Repetowicz, S. Hutzler, and P. Richmond, arXiv:cond-mat/0407770