Turnover Activity in Wealth Portfolios

by

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1 Setting the Stage: A Brief Taxonomy of *Wealth*

1.1 Economic Sources of Wealth

- Income, inheritance, revaluation of assets or liabilities, (human capital?)
- *Savings* are a theoretical accounting tool, essentially describing the mediation from income flows to the stock of wealth

1.2 Economic Uses of Wealth $\Longrightarrow$ Composition of Wealth

- Stakes in private or unincorporated businesses (20%)
- Financial assets like bonds, stocks, and their derivatives (15%)
- Real estate
  - Investment purpose (15%)
  - Owner-occupied housing (30%)
- Retirement provisions and total deposits (15%)

1.3 Stylized Facts

1.3.1 Composition of Wealth

- Over time, the composition of portfolios by wealth class, as well as the overall composition of wealth, remain very stable.
- Composition of wealth portfolios differs markedly among wealth classes.
  - Richest 10% of families own 80% of all nonresidential real estate, 91% of all business assets, 85% of all stock, and 94% of all bonds.
  - Bottom 90% own 64% of principal residences, 55% of life insurance plans, 38% of value of pension accounts, and 40% of deposits.

1.3.2 Functional Form of Statistical Regularity

- Bottom 95% with positive wealth are exponentially or Gamma distributed.
- Top 5% obey a power law.

2 Concept of Statistical Equilibrium

- Market economies consist of a large number of heterogeneous agents
- whose interactions produce aggregate consequences
  - possibly unintended and regularly unforseen
  - interactions feed back into agents' behavior and the environment they interact in
- Vast amount of information in such a complex system does not allow for explanation of distribution by tracing microscopic fate of all agents

Instead, statistical equilibrium describes the statistical properties of aggregate outcomes as a **probability distribution of economic agents over possible outcomes**.

3 Maximum Entropy Principle

- Based on the premise of incorporating solely knowledge that has been given to us and scrupulously avoiding probabilistic statements that imply more information than we actually have
- Virtually all known distributions—discrete as well as continuous—can be derived from the maximum entropy principle
- Maximum entropy principle derives probability distributions from known moment constraints
- Distribution of maximum entropy is ‘most likely’ in combinatorial sense of being achievable in the largest number of ways
- *Concentration theorem*: Overwhelming majority of possible distributions compatible with constraints will have entropy very close to the maximum $\Longrightarrow$ extreme improbability that constraints other than those implied by the maximum entropy principle are responsible for the observed outcome

4 Theory of the Tail Distribution

Conceptualize the economy as a set $\mathbf{K} = \{1, \ldots, K\}$ of *investment opportunities*. For all $k \in \mathbf{K}$, let $V^k(t)$ denote value of economic activity k at time t, and for all $h \in \{1, \ldots, n\}$, $n < \infty$, let $a_h^k(t)$ denote the *position* of household h in activity k. Then the value of wealth portfolio h, denoted $w_h(t)$, follows from the household's combination of the K different investment opportunities:

$$w_h(t) \equiv \sum_{k \in \mathbf{K}} a_h^k(t) V^k(t) \qquad \forall h \in \{1, \ldots, n\}$$

Changes in value of a household's portfolio are either due to

- revaluation of economic activities, or
- changes in the behavior of the household.

Statistical equilibrium **does not** put forward specific theory of portfolio choice and/or asset pricing. Instead, we start from a well-defined macroscopic average—the **growth rate of wealth**.

- We can express the average growth rate as a logarithmic mean, but logarithmic mean has no time dimension. Therefore, introduce two different time scales
  - passage of calendar time, and
  - passage of turnovers.
- With continuously compounded returns, the value of a wealth portfolio at calendar time t will be $w(t) = w(0)\ \exp(Rt)$, where R is the *average return over the calendar time interval* $[0, t]$. The value of the portfolio at turnover time n will be $w(n) = w(0)\ \exp(rn)$, where r is the *average return over the turnover time interval* $[0, n]$. As the respective rates of return are defined as averages, it must be true that at the end of both time scales, $t = T$ and $n = N$ (where T and N are reasonably large and thus coincide), the value of the portfolio will be equal such that

$$RT = rN.$$

- Rates of return are proportional to wealth levels if at the beginning of the turnover process all agents start with the same wealth $w(0) = w_0$. Then we can go from the distribution of returns per time scale to the distribution of wealth and vice versa.
- **GOAL:** Obtain an indicator for the rate of turnovers per calendar time. But any information on turnovers that would allow to identify their absolute number ($\mathrm{e.g.}$ distribution of returns per turnover) is unobservable. Thus the strategy will be to obtain an indicator for turnover activity from observed information on wealth and returns per calendar time that tells us about *relative* changes in turnover activity.
- **Step 1** Derive the distribution of returns per turnover from maximum entropy principle (MaxEnt). Let $\bar{r}$ and $\bar{R}$ designate the sample moments of rates of return per turnover and calendar time. Though returns per turnover are unobservable, we can take advantage of the relationship $\bar{r} = \bar{R}T/N$ because $\bar{R}$ is observable.

- MaxEnt program to derive distribution of r reads

$$\max_{f(r)} H \equiv -\int_Z \mathrm{d}r\, f(r) \log f(r)$$

subject to

$$\int_Z \mathrm{d}r\, f(r)\, r = \bar{R}\,T/N,$$
$$\int_Z \mathrm{d}r\, f(r) = 1\,,$$

where we want to allow for negative rates of return and thus define the support $Z = [r_m, \infty)$, and r_m designates the (possibly negative) minimum return per turnover among the portfolios.

- MaxEnt density under the arithmetic mean constraint is an exponential distribution of the form

$$f(r) = \lambda \mathrm{e}^{-\lambda(r-r_m)}.$$

- **Step 2** Solving for the parameter λ of the exponential distribution, we get $\lambda = 1/(\bar{r} - r_m)$. Since the least successful portfolio achieves the smallest average return per turnover r_m and thus also the lowest average return per calendar time R_m, we have that $Nr_m = TR_m$ and

$$\lambda = \frac{N}{T\left(\bar{R} - R_m\right)}.$$

- **Step 3** Corresponding distribution of wealth levels is obtained from the theorem of densities of a function of a random variable, yielding

$$f(w) = \phi w_m^{\phi}\, w^{-(\phi+1)},$$

where ϕ is the **characteristic exponent** of the power law distribution of wealth that obeys

$$\phi = \frac{\lambda}{N}.$$

- **Final Step** Putting all results together, we obtain our indicator for turnover activity as

$$\frac{1}{T} = \phi\left(\bar{R} - R_m\right).$$

- Notice that
  - N has disappeared,
  - which is actually what we expected—it would seem very odd to recover the absolute number of turnovers without having any information on turnover activity in the first place.
  - Instead our indicator of turnover activity measures changes in the portfolio compositions **relative** to the chosen calendar time scale T.

Summary of Power Law Theory

- Statistical equilibrium wealth distribution defines probability field over **returns** from available combinations of investment opportunities such that the growth constraint is met.
- Most decentralized investment activity of agents forms the conceptual basis of **maximum entropy** distribution of wealth.
- Entropy formalism "hesitates" to assign an enormously large return factor to a portfolio because it thereby reduces degrees of freedom in the remaining assignments of return factors that have to meet the growth constraint.
- Statistical equilibrium does not exclude the possibility of extreme outcomes, it merely attaches very low probability $\longrightarrow$ **power law distribution**.
- Essential to obtaining power law as statistical equilibrium outcome: maximum entropy program contains a constraint on the **logarithmic mean**.

5 Empirical Calibration

5.1 The Data: *Forbes 400* List

- To be considered for list, individuals must be US citizens and own more than $\$550$ million at time of annual compilation.
- Uses publicly available information to determine personal wealth from
  - public stock holdings (multiplying share prices with the number of shares known to be in agents' possession),
  - real estate (publicly known square feet-ownership multiplied by local market rates),
  - unincorporated businesses (assumes enterprise operates under same margins as a publicly traded company in the same sector).

  $\Longrightarrow$ Projected tax liabilities are subtracted to obtain an estimate of net worth.

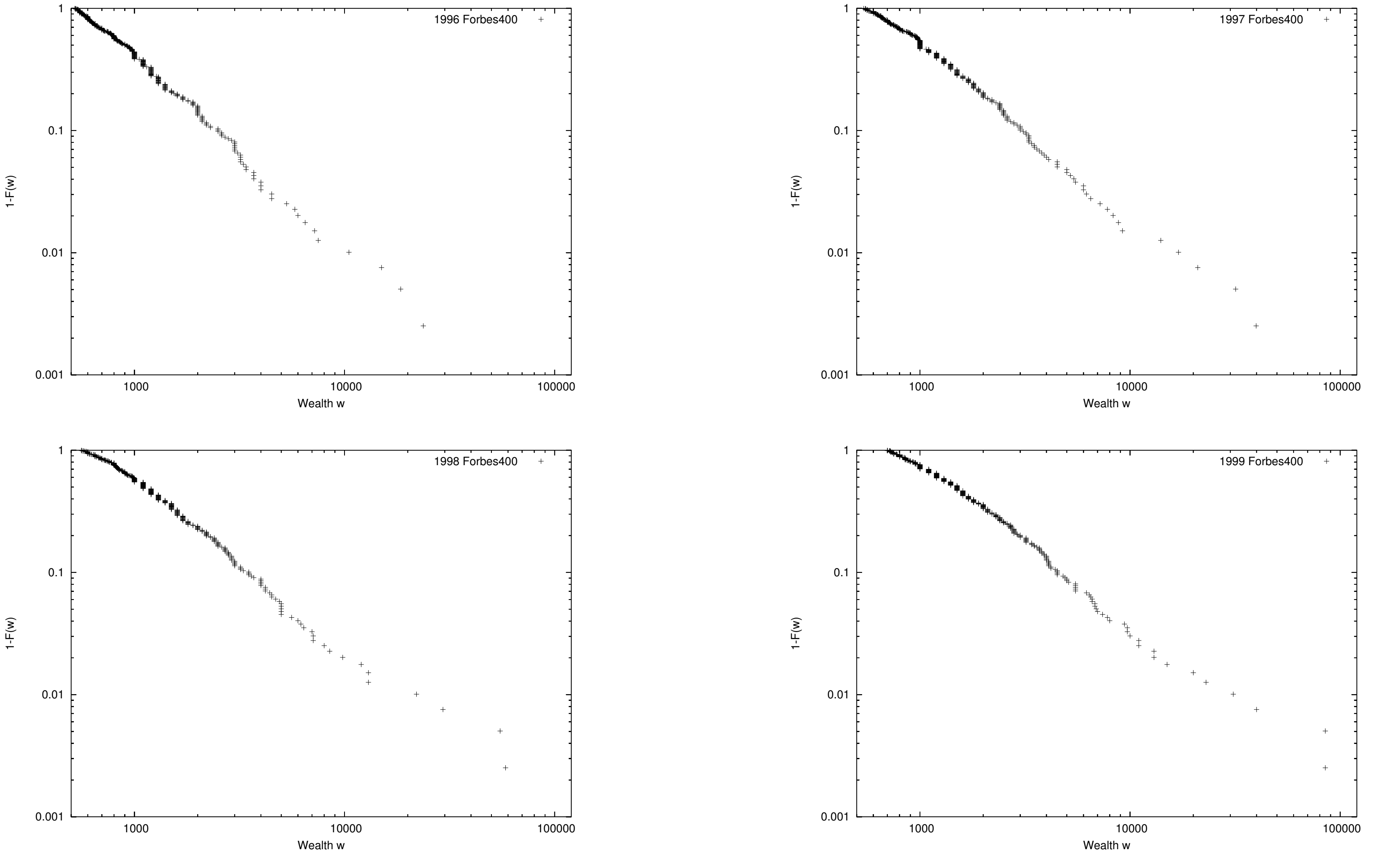

Figure 1: Inverse cumulative distribution of personal wealth form the *Forbes 400* list, plotted on double-logarithmic scale for each of the years 1996–1999.

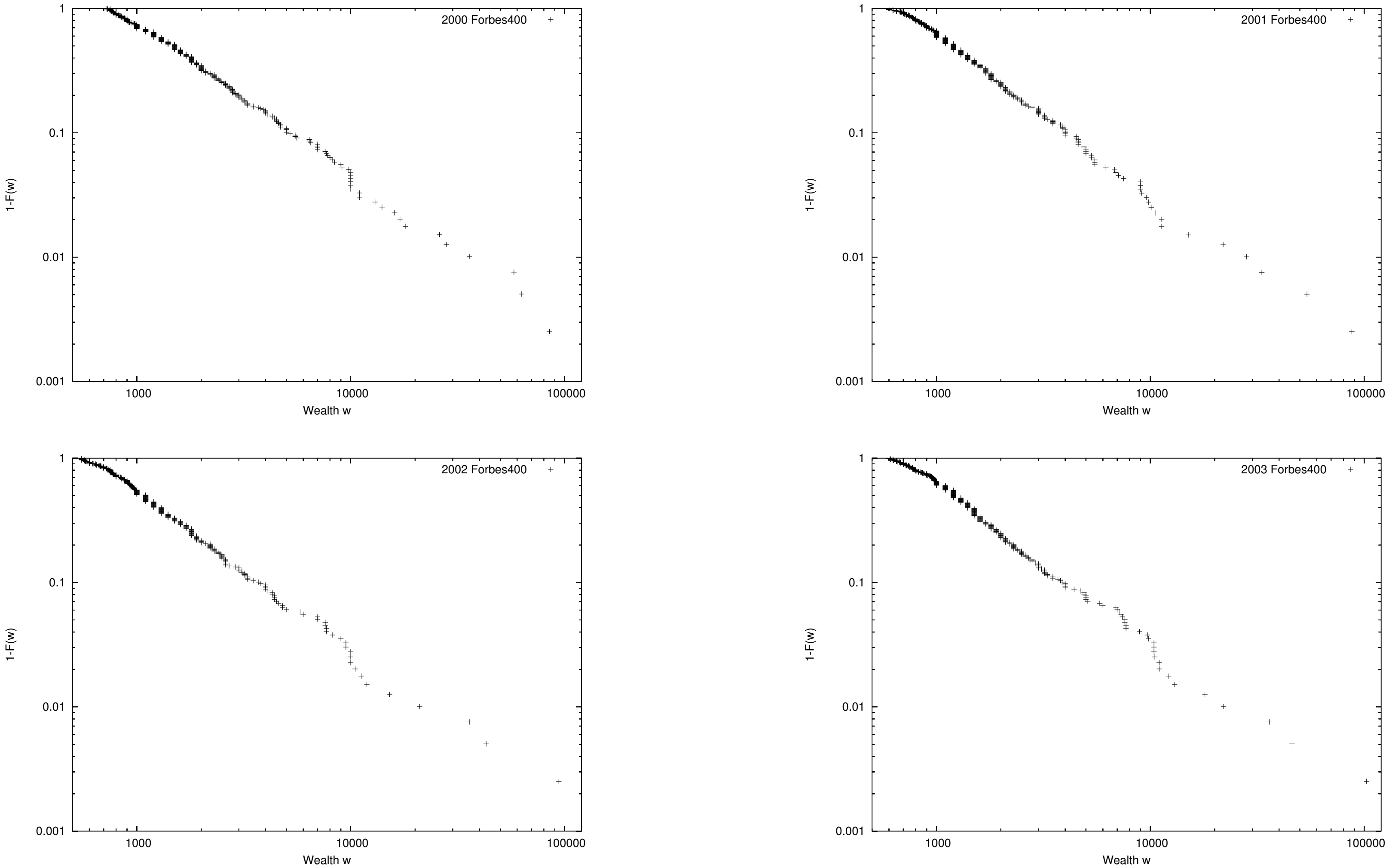

Figure 2: Inverse cumulative distribution of personal wealth form the *Forbes 400* list, plotted on double-logarithmic scale for each of the years 2000–2003.

5.2 Estimates of Characteristic Exponent

Year	1996	1997	1998	1999	2000	2001	2002	2003
$\hat{\phi}$	1.573	1.497	1.369	1.315	1.262	1.358	1.341	1.347
	(.100)	(.095)	(.087)	(.083)	(.080)	(.086)	(.085)	(.085)

Table 1: Characteristic exponent and standard error, top 250 observations of the *Forbes 400*. Standard errors reported in parentheses.

Magnitude of $\hat{\phi}$ is also a direct measure of **inequality** within the power law distribution, with a higher (absolute) $\hat{\phi}$ implying a more equal distribution of wealth.

5.3 Inequality, Turnover Activity, and Mobility

- Estimation of power law exponent completes the first step of calibration.
- Now we need to calculate the growth rate of wealth per year among the wealthy subset, $\bar{R}$, as well as the minimum return per year, R_m, since the turnover indicator is given by $\frac{1}{T} = \phi\left(\bar{R} - R_m\right)$.
  What we calculate, in fact, are empirically observed returns per one calendar year, $\tilde{R}_t = \log w_t - \log w_{t-1}$ that we use as a proxy.
- To make full use of the data, we also calculate a simple measure of mobility by calculating the average absolute change in rank (along with entries and exits and frequency tables for the number of years that agents stay in the lists).

- Generally positive correlation between growth, turnovers, and mobility but
- no clear relationship between those series and inequality.

Year	$\hat{\phi}$	$\bar{R}$	R_m	$1/T$	Mobility
1996-97	1.497	0.191	-0.901	1.635	32.77
1997-98	1.369	0.092	-0.630	0.988	31.18
1998-99	1.315	0.234	-0.642	1.151	35.81
1999-00	1.262	0.090	-0.941	1.302	33.28
2000-01	1.358	-0.129	-2.342	3.004	41.47
2001-02	1.341	-0.087	-1.150	1.425	32.78
2002-03	1.347	0.102	-0.343	0.599	25.63

Table 2: Inequality, average and minimum returns, turnover indicator and rank mobility, *Forbes 400* list.

6 Empirical Distribution of Returns

- The model assumes that there is a period where all agents start with $w_h(0) = w_0 \quad \forall h$,
- but in practice we observe empirical returns $\tilde{R}_t$ between points in time where the distribution is already a power law.
- If theoretical returns are exponentially distributed because they are proportional to the logarithm of wealth, $\tilde{R}_t = \log w_t - \log w_{t-1}$ can be thought of as the difference of two random variables that are in principle exponentially distributed, which should result in an (asymmetric) Laplace distribution.
- On semi-logarithmic scale, the Laplace distribution has a tent shape.

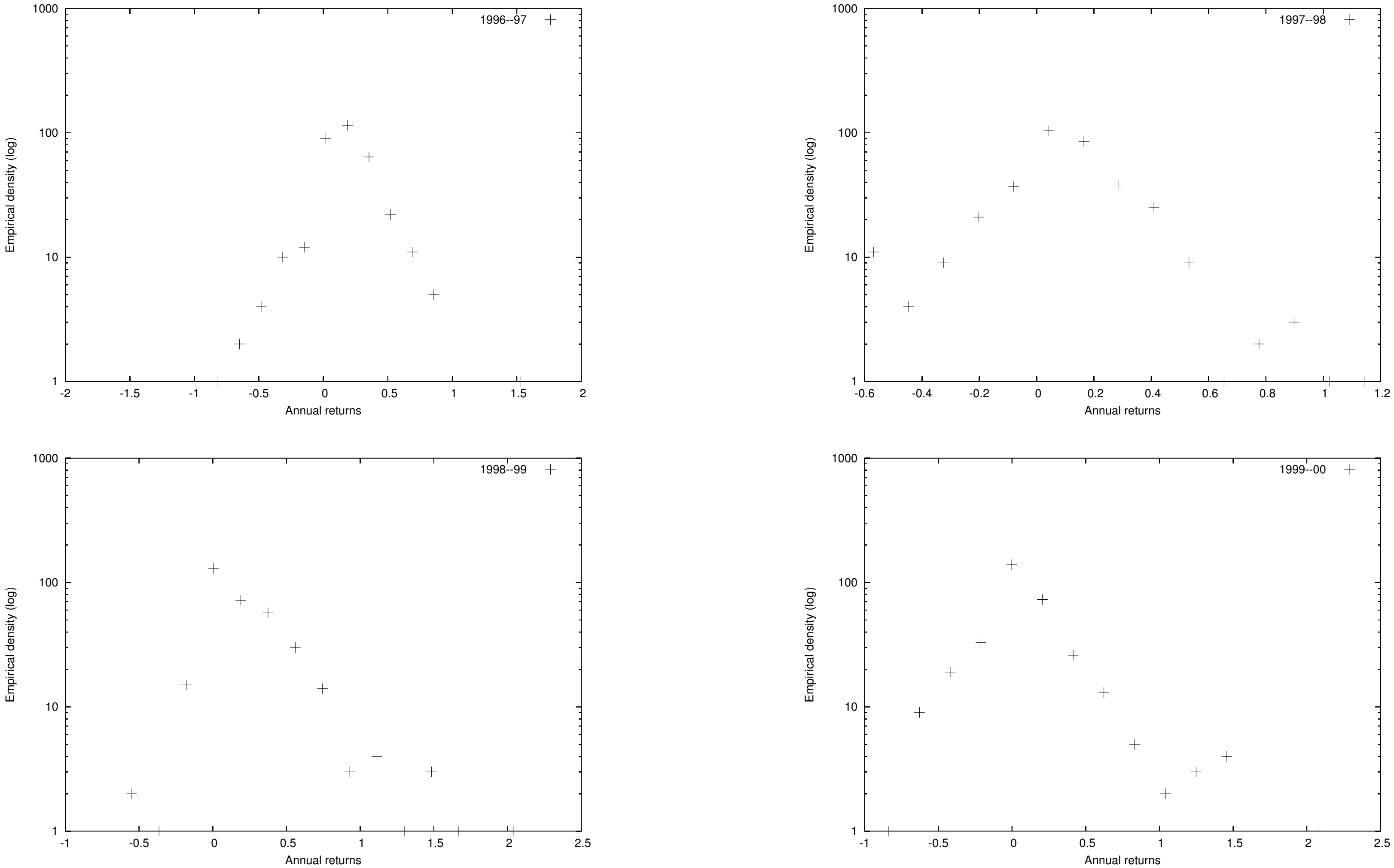

Figure 3: Empirical density (15 bins) of annual returns $\tilde{R}_t$ from the *Forbes 400* list, plotted on semi-logarithmic scale for each of the years 1997–2000.

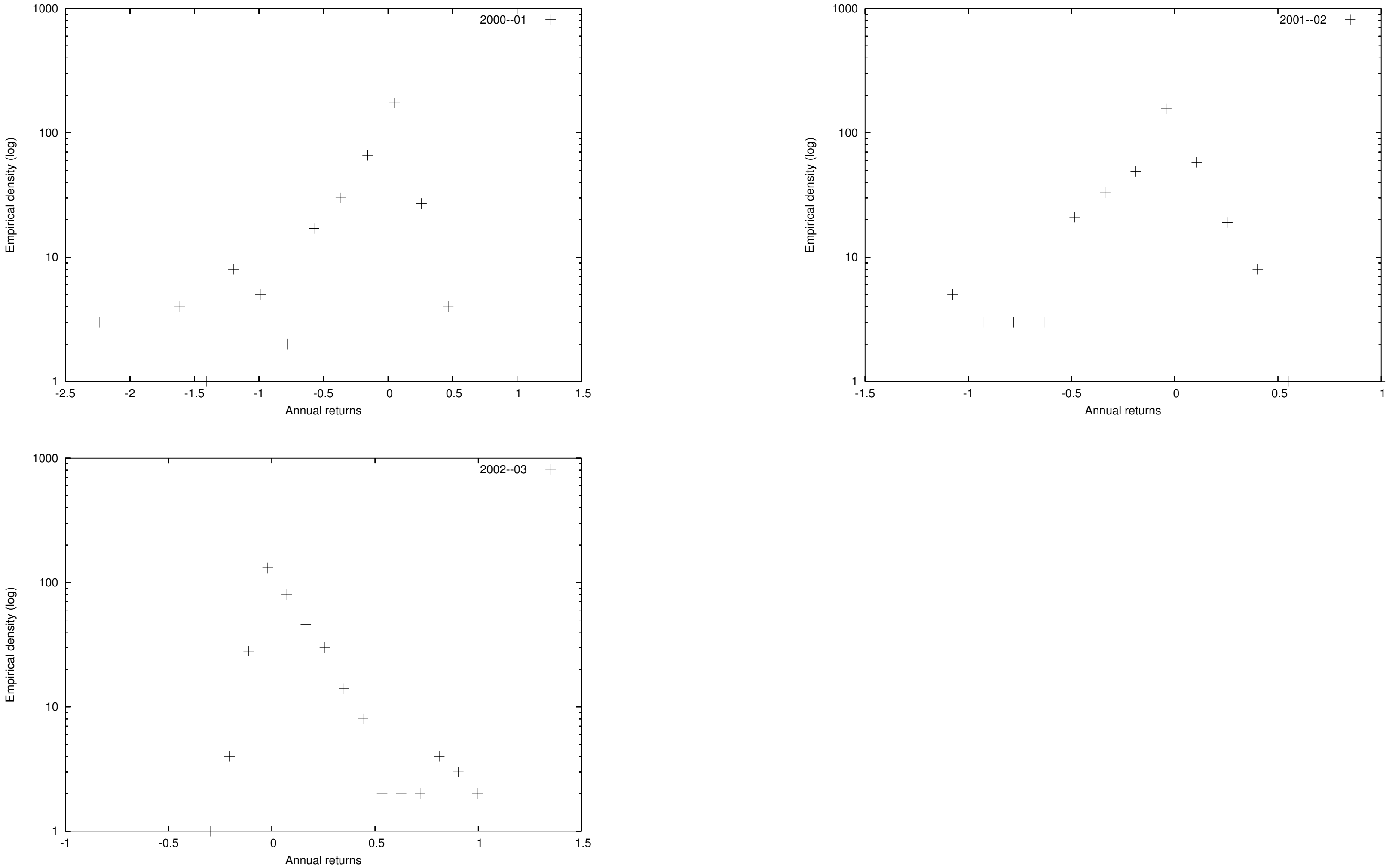

Figure 4: Empirical density (15 bins) of annual returns $\tilde{R}_t$ from the *Forbes 400* list, plotted on semi-logarithmic scale for each of the years 2001–2003.

7 Summary of Results

- Statistical equilibrium model accounts for power law distribution of wealth,
- and is in line with the (asymmetric) Laplace distribution of returns.
- The data allow a calibration of the model, with the aim of obtaining an indicator for turnover activity.
- Turnovers are positively correlated with mobility and (absolute) growth rates,
- but inequality does not show a clear pattern with any of the other variables.
- The results are most likely biased due to sample truncation to the left. Ideally, we would like to have data on the entire power law tail and not just the small subsets we examined.